MODELING UNEQUALLY SPACED 2-D DISCRETE SIGNALS BY RATIONAL FUNCTIONS

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ABSTRACT

A new 2-D interpolating continued fraction is described. This continued fraction is used to model 2-D discrete signals by bivariate rational functions. An algorithm for the recursive determination of the model parameters is presented. Attractive characteristics of the proposed method are applicability to unequally - spaced data and permanence of the solution. The method is applicable to rectangular arrays of data and can be used in a variety of applications, including the design of 2-D IIR digital filters. Generalization and other areas of applications are mentioned.

INTRODUCTION

Interpolation methods, both polynomial and rational, 1-D and n-D, have played a significant role in many areas of signal and system studies. Recently they have been applied to multidimensional signal reconstruction problems [1] and system realization problems [2].

Polynomial interpolation methods, have also been used for modeling 1-D and 2-D signals [3,4]. Frequency sampling FIR filter designs, using the DFT and therefore uniform samples, are essentially interpolation methods on the unit circle [5]. Extensions to nonuniform samples, for both the 1-D and 2-D FIR cases, were published recently [6,7]. The following questions arise: 1) how can we develop interpolation methods for modelling 2-D discrete signals by rational functions and 2) how should these methods be modified in order to serve the design of IIR 2-D frequency sampling filters.

This paper is concerned with the problem of the first question and proposes a method for its solution. The method is based on a new 2-D interpolating continued fraction which has the following attractive features:

1) applicability to nonuniform samples
2) recursive computation of the model parameters
3) permanence (if we want to use more data to compute a higher order model, we compute only the "new" coefficients)

The structure of the paper is as follows: Under preliminaries we review some basic facts about 1-D rational interpolation and interpolating continued fractions (CF), to be used in the sequel. A statement of the problem treated in the paper, with remarks referring to characteristics of the problem and methods of solution, follows. The next section describes a new 2-D interpolating CF and its properties. Construction and use of this CF constitute the original contribution of the paper. We continue with a description of the proposed modeling algorithm for a rectangular array of data. The last section contains some remarks about the modeling method used, conclusions and suggestions for further work in this area.

PRELIMINARIES

We first review some basic facts about 1-D polynomial and rational interpolation. These facts will be needed in the sequel.

The well known 1-D interpolation problem [9] can be solved recursively using Newton's method. This method rests on the theorem:
"every polynomial of degree n

$$p_n(x) = \alpha_0+\alpha_1 x+\alpha_2 x^2+\dots+\alpha_n x^n \tag{1}$$

can be written uniquely in the form

$$p_n(x) = c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)+\dots$$
$$\dots+c_n(c-c_0)\dots(x-x_{n-1})\text{"} \tag{2}$$

This is called Newton's form

The 1-D rational interpolation problem can be stated as follows [9,10]:
Given 1) a set of points

$$S = \left[x_i \ : \ i=0,1,\dots,M-1\right]$$

2) a set of values at those points

$$S_f \boxdot \left[f(x_i) \boxdot f_i \ : \ i=0,1,\dots,M-1\right]$$

3) a rational function of the form

$$r(x) = \frac{p_n(x)}{q_m(x)} = \frac{p_0+p_1x+\dots p_nx^n}{1+q_1x+\dots+q_mx^m}, \quad m+n=M-1 \tag{3}$$

find p_i , $i=0,1,\dots,n$ and q_j , $j=1,\dots,m$
such that

$$r(x_i) = f_i \quad i=0,1,\dots,M-1 \text{ (interpolating conditions)} \tag{4}$$

It is well known that there are certain difficulties associated with this problem and several algorithms for computing its solution are available [9,10]. Desirable properties of a method of solution are a) existence b) uniqueness c) recursive computation of the unknowns d) permanence and e) applicability for not equidistant interpolation points.

A method of solving the rational interpolation, which has the above characteristics, is based on continued fractions (CF) [9,10]. It is known that a CF of the form

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$$K_M(x) = b_0 + \cfrac{x-x_0}{b_1 + \cfrac{x-x_1}{b_2 + \cfrac{x-x_2}{\ddots + \cfrac{x-x_{M-2}}{b_{M-1}}}}} =$$

$$= \frac{P[\frac{M}{2}](x)}{q[\frac{M-1}{2}](x)} \qquad (5)$$

known as Thiele's CF, interpolates at the points x_i and its unknown's b_j can be computed recursively. Furthermore (5) is a rational function with numerator and denominator polynomials of known degrees $[\frac{M}{2}]$ $[\frac{M-1}{2}]$ respectively.

STATEMENT OF THE PROBLEM

The approach to modeling taken here is that of rational interpolation with unequally spaced samples. The 2-D rational interpolation problem, in a form similar to the 1-D problem mentioned in the preliminaries, can be stated as follows:

Given:

1) an array of points

$$S = \left[(x_i,y_j): \quad i=0,1,\ldots,M-1; \quad j=0,1,\ldots,N-1\right]$$

2) a set of values at those points

$$S_f = \left[f(x_i,y_j) \triangleq f_{ij}: \quad i=0,1\ldots,M-1; \quad j=0,1,\ldots,N-1\right] \qquad (6)$$

3) a 2-D rational function (model)

$$r(x,y) = \frac{p(x,y)}{q(x,y)} = \frac{p_{00}+p_{10}x+p_{01}y+p_{11}xy+p_{20}x^2+p_{02}y^2+\ldots}{1+q_{10}x+q_{01}y+q_{11}xy+\ldots} \qquad (7)$$

find MxN coefficients of the polynomials p(x,y) and q(x,y) such that the following interpolating conditions are satisfied:

$$r(x_i,y_j) = f_{ij}, \quad i=0,1,\ldots,M-1; \quad j=0,1,\ldots,N-1 \qquad (8)$$

The following remarks are appropriate:

1. Essential elements of the problem are
   a) the form of the interpolation array (here it is rectangular)
   b) the number of interpolation points
   c) the number of unknowns in the interpolating function r(x,y) (the model)
   d) the number and kind of the interpolating conditions
2. Selection must be made of the number of unknowns in the numerator and denominator of r(x,y) (i.e the "degree" and form of the polynomials p(x,y) and q(x,y)).
3. Different choises in the "essential elements of the problem" lead to different difficulties and methods of solution.
4. If we select the number of interpolation points to equal the number of conditions, we expect, in general, to obtain equations whose solution is a solution to the interpolation problem.
5. One approach, usually called "the direct approach" [7], is to solve the equations

$$f_{ij}q(x_i,y_j)-p(x_i,y_j) = 0 \quad \begin{cases} i=0,1,\ldots,M-1 \\ j=0,1,\ldots,N-1 \end{cases} \qquad (9)$$

   This approach presents difficulties even in the 1-D case [10].
6. A better approach is to look for a method of solution with the following desirable properties easily judged:
   a) existence
   b) uniqueness
   c) recursive computability
   d) permanence

The method of interpolating CF, in the 1-D case, has those properties. We seek therefore a 2-D interpolating CF which preserves the properties of the 1-D case. This we do in the next section.

THE METHOD OF 2-D INTERPOLATING CF

The main idea of this paper, in extending rational interpolation to 2-D, is to use a CF of the form

$$K_{MN}(x,y) = \frac{C_{00}}{1+h_0(x)+h_1(x,y)+h_2(x,y)+\ldots+h_{N-1}(x,y)} \qquad (10)$$

where

$$h_0(x) = \cfrac{C_{10}(x-x_0)}{1+\cfrac{C_{20}(x-x_1)}{1+\ddots + C_{M-1,0}(x-x_{M-2})}} \quad ,$$

$$h_k(x,y) = \cfrac{C_{0k}(y-y_0)(y-y_1)\ldots(y-y_{k-1})}{1+\cfrac{C_{1k}(x-x_0)}{1+\cfrac{C_{2k}(x-x_1)}{1+\ddots+C_{M-1,k}(x-x_{M-2})}}} \qquad (11)$$

$$(k=1,2,\ldots,N-1)$$

Imposing the interpolating conditions,

$$K_{MN}(x_0,y_j) = f_{ij}, \quad i=0,1,\ldots,M-1; \; j=0,1,\ldots,N-1 \qquad (12)$$

in a specified order, we can obtain the coefficients C_{ij} successively. This method of solution, not only gives the coefficients recursively, but also has the so called permanence property. This means that if with a certain amount of data f_{ij} we determine the corresponding C_{ij}, and then with more data we want to determine more C_{ij}'s, we only have to determine the new coefficients, leaving the ones already computed unchanged. Existence and uniqueness of the C_{ij} presupose distinct x_i's and y_j's and the $f_{ij} \neq 0$ whenever they appear in denominators.

From (10) and (11), by proper choices of the variables we obtain the following special cases:

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$$K_{M0}(x,y_0) = \cfrac{C_{00}}{1+\cfrac{C_{10}(x-x_0)}{1+\cfrac{C_{20}(x-x_1)}{1+\ddots+C_{M-1,0}(x-x_{M-2})}}} \qquad (13)$$

and

$$K_{0N}(x_0,y) = \frac{C_{00}}{1+C_{01}(y-y_0)+\ldots+C_{0,N-1}(y-y_0)\ldots(y-y_{N-2})} \qquad (14)$$

(13) is a Thiele-type interpolating CF (compare with (5)) and (14) is C_{00} divided by a Newton Polynomial (compare with (2)). This CF, as can be seen from its structure, combines ideas of 1-D polynomial and rational interpolation [9,10] to interpolate on a rectangular array of data. The interpolation problems on square and triangular arrays are special cases of (10) and (11). The way this CF is composed, from 1-D Newton interpolating polynomial and Thiele's 1-D CF, suggests an algorithm for computing the C_{ij}. This we present in the next section.

THE PROPOSED MODELING ALGORITHM

To simplify the presentation we describe the algorithm for computing recursively the C_{ij}'s in (11), for the case M=4 and N=3. The generalization for arbitrary values of M and N is straightforward. We have then from (10)

$$K_{43}(x,y) = \frac{C_{00}}{1+h_0(x)+h_1(x,y)+h_2(x,y)} \qquad (15)$$

where

$$h_0(x) = \cfrac{C_{10}(x-x_0)}{1+\cfrac{C_{20}(x-x_1)}{1+C_{30}(x-x_2)}}, \quad h_1(x,y) = \cfrac{C_{01}(y-y_0)}{1+\cfrac{C_{11}(x-x_0)}{1+\cfrac{C_{21}(x-x_1)}{1+C_{31}(x-x_2)}}},$$

$$h_2(x,y) = \cfrac{C_{02}(y-y_0)(y-y_1)}{1+\cfrac{C_{12}(x-x_0)}{1+\cfrac{C_{22}(x-x_1)}{1+C_{32}(x-x_2)}}} \qquad (16)$$

From (12), in this case, we obtain

$$K_{43}(x_i,y_j) = f_{ij} \qquad i=0,1,2,3; \quad j=0,1,2 \qquad (17)$$

The algorithm then, from (15) and (17), is as follows:

step 1: $C_{00}=K_{43}(x_0,y_0)=f_{00}$ (18)

step 2: $K_{43}(x,y_0)=f_{10}$ and

$$C_{10} = \frac{\frac{C_{00}}{f_{10}}-1}{x_1-x_0}, \quad f_{10}\neq 0 \qquad (19)$$

step 3: $K_{43}(x_2,y_0) = f_{20}$ and $C_{20}=\dfrac{\dfrac{C_{10}(x_2-x_0)}{\frac{C_{00}}{f_{20}}-1}-1}{x_2-x_1}$, $f_{20}\neq 0$ (20)

step 4: $K_{43}(x_3,y_0)=f_{30}$ and $C_{30} = \dfrac{\dfrac{C_{20}(x_3-x_1)}{\dfrac{C_{10}(x_3-x_0)}{\frac{C_{00}}{f_{30}}-1}-1}-1}{x_3-x_2}$, $f_{30}\neq 0$ (21)

step 5: Similarly from

$$K_{43}(x_i,y_1) = f_{i1} = \frac{C_{00}}{1+h_0(x_i)+h_1(x_i,y_1)}, \quad i=0,1,2,3 \qquad (22)$$

(i.e using the second row of data points) we obtain successively C_{i1}, i=0,1,2,3, and

step 6: from

$$K_{43}(x_i,y_2) = f_{i2} = \frac{C_{00}}{1+h_0(x_i)+h_1(x_i,y_2)+h_2(x_i,y_2)} \qquad (23)$$

we obtain successively C_{i2}, i=0,1,2,3.

If the model obtained is desired to be in rational function form, we must convert the CF (10) and (11) to rational function. Concerning the form of the numerator and denominator polynomials, as functions of the M and N in (10), and a conversion algorithm see [11].

FINAL REMARKS AND CONCLUSIONS

We described a new 2-D interpolating CF and used it to model 2-D discrete signals when the data are given on a rectangular array. The form of the CF is such that other arrays, such as triangular, hexegonal etc. are special cases of the one used. Extension to higher dimensional cases is straightforward, in principle. By applying the interpolation conditions on the selected CF, in a particular order, we obtained the following properties of the sulution of the modeling problem a) existense b) uniqueness c) recursive computability d) permanence. The method is applicable to non-equidistant interpolation points and can be used in a variety of applications such as the estimation of derivatives, integrals, missing values, image processing etc. In particular if we let $x=z_1^{-1}$ and $y=z_2^{-1}$ in the CF we can use the proposed method to design IIR frequency sampling filters. Work is in progress in this area and will be reported. Other forms of 2-D interpolating CF, which can also be used for signal modeling are described in [8]. Future work could include

a) a comparison of various 2-D CF

b) use of Hermite-type interpolating conditions (derivative data are then required)
c) digital network structures which correspond to the 2-D CF
d) connection of the CF with 2-D Padé approximants
e) connection wtih 2-D moments and orthogonal expansions
f) application of the 2-D CF to system reduction problems
g) algorithms for reducing the CF to rational form.

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