FREQUENCY SAMPLING DESIGN OF 2-D IIR FILTERS USING CONTINUED FRACTIONS

John E. Diamessis, Gerasimos G. Potamianos
National Technical University of Athens
Patission St. 42, Athens 106 82, Greece

ABSTRACT

A method of designing two-dimensional IIR filters using nonuniform frequency samples, is presented. The method is based on an interpolating 2-D continued fraction which has a number of desirable properties: recursive computation of the design parameters, permanence and reality of the resulting transfer function. An example is given and areas for future research are mentioned.

I. INTRODUCTION

Frequency sampling FIR filter designs have received consederable attention both in the one and the two dimensional cases [1,2,8]. They are usually based on the DFT and they are, therefore, limited to uniform frequency samples. Extensions to nonuniform samples, for both the 1-D and 2-D FIR cases, were published recently [3, 4,5]. The core of these methods, for FIR filters, is polynomial interpolation on the unit circles of the complex plane. It follows then, that for the corresponding IIR designs we need rational interpolation methods on the unit circles. It is known that the solution of rational interpolation problems presents some difficulties, and to use them for filter design, we face the added problem of obtaining a real transfer function (although our data are complex numbers).

This paper presents a method of designing 2-D IIR filter using nonuniform or uniform samples of the frequency response. The proposed design procedure is based on an interpolating continued fraction form of the transfer function [6,7]. Important properties of this continued fraction are:

1) gives a real rational function, although the data are complex
2) leads to an algorithm, which computes recursively the design parameters (i.e. the fraction's coefficients)
3) partial or total permanence depending on the structure of the chosen continued fraction [if we want to use more data to obtain a higher order filter, we only need to compute the "new" (additional) parameters]
4) can be used with uniform or nonuniform frequency samples.

The proposed method can also be used in conjuction with optimization methods of filter design in two ways:

1) use the design parameter vector, obtained by interpolation, as the initial "point" in the optimization method
2) obtain part of the design parameters by interpolation and the rest by same optimization criterion (in order to "fit" better).

The design method does not secure a stable filter. However, a separate treatment of the approximation and the stability problem, for 2-D systems, seems to be the present state of the art [8].

II. STATEMENT OF THE PROBLEM

The problem of the frequency sampling design of 2-D IIR filters can be stated as follows:

Given:

1) a set of sampling points

$$S \triangleq \{(w_k, z\ ):\ k=0,1,\dots,m-1\ \ \ell=0,1\dots,n-1\}$$

on the unit circles $|w| = 1$, $|z| = 1$

where $w_k = \exp(j\phi_k)$, $\phi_k\ [0,\pi]$, $k=0,1,\dots$

$z = \exp(j\vartheta\ )$, $\vartheta\ [0,\pi]$, $\ell=0,1,\dots$

2) A matrix of the desired frequency response values $H_{k\ell}$ and $H_{k,-\ell}$ at the sampling points (ϕ_k,ϑ_ℓ) and $(\phi_k,-\vartheta_\ell)$ respectively. (These values are in general complex).

find: the transfer function $G(w,z)$ of a 2-D IIR digital filter which has the desired frequency response values at the points of the "sampling grid".

The problem then is to find $G(w,z)$ such that

$$G(w_k,z_\ell) = H_{k,\ell} \tag{1}$$

and

$$G(w_k,z_{-\ell}) = H_{k,-\ell} \begin{cases} k=0,1\dots,m-1 \\ \ell=0,1,\dots,n-1 \end{cases} \tag{1α}$$

Note that this problem, as stated, is essentially a 2-D rational interpolation problem, and that the given frequency samples are not required to be uniformely spaced. It can be seen that 1-D frequency sampling IIR design problem is a special case of the problem stated above. This case is treated fully in [7]. Both in the 1-D and 2-D cases interpolating continued fraction (ICF) offer definite theoretical (existence, uniqueness) and computational advantages when used to solve rational interpolation problems. In addition the ICF proposed here can inverted to give a real transfer function and so to satisfy ralization requirements of the filter.

III. 2-D INTERPOLATING CONTINUED FRACTIONS (I.C.F.)

The solution of the filter design problem stated in the previous section is obtained by a method which can be called "the method of I.C.F.". The main idea is to use a 2-D continued fraction which not only interpolates at the sampling frequencies, but also has a structure which reduces to a real rational function, in spite of the fact that the data are complex numbers.

To make the presentation smoother, we cover under

CH2868-8/90/0000-2454$1.00

preliminaries, below, two auxilary topics: a) the solution of the 2-D Cauchy-type interpolation problem with real data (using CFs) and b) a theorem which containes the key elements of the main idea of the **proposed filter design** method (for the 1-D case). A combination and generalization of a) and b) leads in a clear manner to the 2-D filter design method.

A. PRILIMINARIES

1. The 2-D Cauchy-type Interpolation Problem [6]

This problem can be stated as follows:

Given:

1) a rectangular grid of mxn points determined by the sets

$$\{x_0 x_1,\ldots,x_{m-1}\} \quad \text{and} \quad \{y_0,y_1,\ldots,y_{n-1}\}$$

2) a set of numbers f_{ij} , $\begin{cases} i=0,1,\ldots,m-1 \\ j=0,1,\ldots,n-1 \end{cases}$ (1)

Find a 2-D rational function $r(x,y)$ which satisfies the conditions

$$r(x_i,y_j) = f_{ij} \;,\; \begin{cases} i=0,1,\ldots,m-1 \\ j=0,1,\ldots,n-1 \end{cases} \tag{2}$$

One form of a CF which solves this problem is given in [6]. It is shown there that the coefficients of the CF can be computed recursively and that they are permanent. These are highly desirable characteristics both computationally and when these coefficients play the role of design parameters (as will be shown in the sequel).

Onother form of a CF which also satisfies (2), and which we will use in this paper, is the following:

$$r(x,y) \triangleq A_0 + \cfrac{x-x_0}{A_1 + \cfrac{x-x_1}{A_2 + \ddots + \cfrac{x-x_{m-2}}{A_{m-1}}}} \tag{3}$$

where

$$A_i \quad A_i(y) \triangleq \alpha_{i0} + \cfrac{y-y_0}{\alpha_{i1} + \cfrac{y-y_1}{\alpha_{i2} + \ddots + \cfrac{y-y_{n-2}}{\alpha_{i,n-1}}}} \tag{4}$$

(i 0,1,...,m-1)

2. 1-D Rational Complex Interpolation [7]

The key idea of the proposed design method, in the 1-D case, is contained in the following theorem. For simplicity we consider here a special case: Suppose we are given three sampling points $\{z_0,z_1,z_2\}$ $z_k = \exp(j\vartheta_k)$, $z_k \notin \mathbb{R}$ and the desired values of a transfer function $H_d(z_k) \triangleq H_k$, k=0,1,2, at those points. We form the functions

$$G_1(z) \triangleq \cfrac{\alpha_0(z^{-1}-z_0)}{1 + \cfrac{\alpha_1(z^{-1}-z_1)(z^{-1}-z_0^*)}{1 + \alpha_2(z^{-1}-z_2)(z^{-1}-z_1^*)}} \tag{5a}$$

$$G_2(z) \triangleq \cfrac{\alpha'(z^{-1}-z_0^*)}{1 + \cfrac{\alpha_1'(z^{-1}-z_1^*)(z^{-1}-z_0)}{1 + \alpha_2'(z^{-1}-z_2^*)(z^{-1}-z_1)}} \tag{5b}$$

and

$$G(z) \triangleq \frac{1}{1 + G_1(z) + G_2(z)} \tag{5c}$$

where * indicates complex conjugate and α_k,α_k' are complex numbers to be determined. It can be shown that:

1) There exist unique α_i which can be obtained recursevely (in terms of the data H_k) from the conditions

$$G(z_k) = H_k \;,\quad 1=0,1,2. \tag{6}$$

2) $\alpha_k' = \alpha_k$, k=0,1,2
3) the α_k are "permanent"
4) $G(z)$ has real coefficients
5) If $G(z) = B(z)/A(z)$, then degA(a) degB(z) and if it is greater, it is so by one.

One the basis of this theorem, we can develope a method [7] by which given H_k, we can find a transfer function $G(z)$ such that

$$G(z_k) = H_k \qquad K=0,1,\ldots,N-1 \tag{7}$$

B. 2-D RATIONAL COMPLEX INTERPOLATION

The 2-D filter design problem stated earlier is a rational complex interpolation problem. To obtain a 2-D real rational function interpolating the complex data we generalize the procedure of A.2, exploiting the following two facts:

a) The values of the frequency response of a filter at (ϕ_k,ϑ_k) and $(-\phi_k,-\vartheta_k)$ are complex conjugate numbers.
b) The sum of two 2-D rational functions with complex conjugate coefficients is a real rational function.

In order to form an ICF, suitable to solve the 2-D filter design problem, we distinguish <u>two cases</u>:

1) All the points on the sampling grid are purely complex numbers.
2) Some sampling points are real (the possible values are w=±1, z=±1)

In the <u>first case</u> the design problem is based on the <u>theorem 1</u>: We form the functions

$$G_1(w,z) \triangleq \cfrac{\alpha_{00}(w^{-1}-w_0^{-1})}{A_0(z) + \cfrac{\alpha_{10}(w^{-1}-w_1^{-1})(w^{-1}-w_0)}{A_1(z) + \cfrac{\alpha_{20}(w^{-1}-w_2^{-1})(w^{-1}-w_1)}{A_2(z) + \ddots}}} \tag{8a}$$

$$G_2(w,z) \triangleq \cfrac{\alpha_{00}'(w^{-1}-w_0)}{B_0(z) + \cfrac{\alpha_{10}'(w^{-1}-w_1)(w^{-1}-w_0^{-1})}{B_1(z) + \cfrac{\alpha_{20}'(w^{-1}-w_2)(w^{-1}-w_1^{-1})}{B_2(z) + \ddots}}} \tag{8b}$$

$$G(w.z) \triangleq \frac{1}{1+G_1(w,z)+G_2(w,z)} \tag{8c}$$

Where

$$A(z) = 1 + \cfrac{\alpha_{\ell 1}(z^{-1}z_1^{-1})}{1 + \cfrac{\alpha_{\ell 2}(z^{-1}-z_1)}{1 + \cfrac{\alpha_{\ell 3}(z^{-1}-z_2^{-1})}{1 + \cfrac{\alpha_{\ell 4}(z^{-1}-z_2)}{1 + \ddots}}}} \tag{9}$$

$$B(z) = 1 + \cfrac{\alpha_{\ell 1}(z^{-1}-z_1)}{1 + \cfrac{\alpha_{\ell 2}(z^{-1}-z_1^{-1})}{1 + \cfrac{\alpha_{\ell 3}(z^{-1}-z_2)}{1 + \cfrac{\alpha_{\ell 4}(z^{-1}-z_2^{-1})}{1 + \ddots}}}} \tag{10}$$

$\ell=0,1,2,\ldots$

It can be shown that: (see the Appendix)

1) There exist unique $\alpha_{k\ell}$ which can be computed recursively (in terms of the data H_k) from the conditions (1).
2) The coefficients $\alpha_{k\ell}$ and $\alpha_{k\ell}$ are complex conjugate numbers.
3) The $\alpha_{k\ell}$ are permanent.
4) G(w,z) has real coefficients.
5) The degrees of the numerator and denominator polynomials of G(w,z) can be determined.

Using this theorem we can solve the 2-D frequency sampling design problem.

Second case: We can state a similar theorem in the case where some of the sampling points are real. We confine our values to a sampling grid formed by the sets $\{0^\circ,\phi_0,\phi_1,\ldots,\phi_{m-1},180^\circ\}$ and $\{0^\circ,\vartheta_0,\vartheta_1,\ldots,\vartheta_{n-1},180^\circ\}$

Theorem 2: We form the I.C.F.

$$G'(w,z) = \cfrac{A}{1+\cfrac{B(w^{-1}+z^{-1}-2)}{1+(z^{-1}+1)H_1(z)+\cfrac{c(w^{-1}-1)}{1+(w^{-1}+1)H_2(w)+\cfrac{D(z^{-1}-1)}{1+\cfrac{w^{-1}+z^{-1}+2}{\frac{1}{H_3(z)}+\cfrac{w^{-1}+1}{\frac{1}{H_4(w)}+C(w,z)}}}}}} \tag{11}$$

where G(w,z) is the 2-D I.C.F of (8) and $H_1(z)$, $H_3(z)$, $H_2(w)$, $H_4(w)$ are 1-D I.C.F of the form:

$$H_i(x) = \cfrac{1}{1+\cfrac{\alpha_{i0}(x-y_0)}{1+\cfrac{\alpha_{i1}(x-y_1)(x-y_0^{-1})}{1+\ddots+\alpha_{i,k-1}(x-y_{k-1})(x-y_{k-2}^{-1})}}+\cfrac{\alpha'_{i0}(x-y_0^{-1})}{1+\cfrac{\alpha'_{i1}(x-y_1^{-1})(x-y_0)}{1+\ddots+\alpha'_{i,k-1}(x-y_{k-1}^{-1})(x-y_{k-2})}}} \tag{12}$$

where for:

$i=1,3$: $x=z^{-1}$, $k=n$, $y_i=z_i$ $i=0,1,\ldots,n$ and for

$i=2,4$: $x=w^{-1}$, $k=m$, $y_i=w_i$ $i=0,1,\ldots,m$.

Conclusions 1),2),4) and 5) of Theorem 1 apply to the I.C.F. (11). The proof is similar. The permanence of the coefficients is now partial.

For Properties of the I.C.F. (12) see A.2 and [7]. Note that A,B,C,D are real numbers and their values are computed at the sampling points $(\phi,\vartheta)=(0^\circ,0^\circ)$, $(0^\circ,180^\circ)$, $(180^\circ,0^\circ)$ and $(180^\circ,180^\circ)$ respectively. At these points the frequency response is real. The coefficients of the I.C.F. (11) (i.e. H_1, H_2, H_3, H_4) are computed at $\psi=0^\circ$, $\vartheta=0^\circ$, $\phi=180^\circ$, $\vartheta=180^\circ$ respectively.

IV. METHOD OF DESIGN

A step by ste description of the proposed method of design is:

i) First we choose the set S which determines the rectangular sampling grid. This choise is crucial and determines the form of the I.C.F. which can give the solution to the problem of §II. The desired frequency response must also be specified at the points of the sampling grid.

ii) Then we compute the uknown coefficients of the I.C.F.

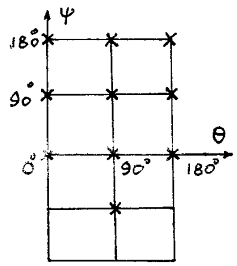
iii) We invert the I.C.F and obtain the transfer function of the filter. (in rational form)

iv) Finally we check the filter for stability. Since the I.C.F's of theorems 1 and 2 cannot quarrantee the stability of the resulting filter this step is necessary. A further step of the stabilization of an unstable filter must be done. Both tasks need a lot of computations for 2-D filters.

An example of the above method (the three first steps) is presented in the following paragraph.

V. AN EXAMPLE

We wish to design a 2-D Band-Pass filter using the ICF (11). We use the following sampling grid (of the half plane (ψ,ϑ)). It consists of 15 points. However, due to symmetries discussed above., we need (as data) the values of the desired frequency response at 10 points: At $(0^\circ,0^\circ)$ the desired amplitude is 0.001 at $(0^\circ,180^\circ)$ and $(180^\circ,0^\circ)$ 10^{-4}, at 0°, 90°) and $(90^\circ,0^\circ)$ is 0.004, at $(90^\circ,180^\circ)$ and $(180^\circ,90^\circ)$ 0,0002, at $(30^\circ,90^\circ)$ and $(90^\circ,90^\circ)$ is 1 and at $(180^\circ,180^\circ)$ is 0.00005.



The phase is 0 at all points. With this data we compute the coefficients of (11). Then we invert the I.C.F. and obtain the following transfer function

$$G(w,z) = \frac{B(w,z)}{A(w,z)} \quad \text{where}$$

$$A(w,z) = a_{00} + a_{10}\,w + a_{01}\,z + a_{20}\,w^2 + a_{11}\,w\,z + a_{02}\,z^2 + \ldots$$

$$B(w,z) = b_{00} + b_{10}\,w + b_{01}\,z + b_{20}\,w^2 + b_{11}\,w\,z + b_{02}\,z^2 + \ldots$$

and

b00 =	6.8845233136E-04	a00 =	6.3256030631E-01
b10 =	2.1471983512E-03	a10 =	1.9365261157%+00
b01 =	4.1291373080E-03	a01 =	3.2822379757E+00
b20 =	1.2223414745E-03	a20 =	1.2015778613E+00
b11 =	3.5312219179E-03	a11 =	7.0012137862E-01
b02 =	1.4527539643E-03	a02 =	1.1798449734E-01
b30 =	2.6321116541E-05	a30 =	9.3333832509E-02
b21 =	5.5121779454E-04	a21 =	4.3032132159E-01
b12 =	2.5859013838E-03	a12 =	1.4123821427E+00
b03 =	-1.0226789390E-03	a03 =	1.8884072578E+00
b40 =	-2.1796747799E-07	a40 =	1.3188820223E-03
b31 =	8.9497561204E-04	a31 =	1.6976903683E+00
b22 =	1.7723098177E-03	a22 =	3.5364935883E+00
b13 =	4.9580302076E-04	a13 =	8.7924372138E-01
b04 =	-7.1981519070E-05	a04 =	-9.6214716387E-01
b50 =	-1.3081779885E-19	a50 =	-1.2511589883E-05
b41 =	1.0062276120E-04	a41 =	1.5999488199E-01
b32 =	1.8959006904E-04	a32 =	-5.4849758127E-01
b23 =	1.2236262709E-03	a23 =	-9.2173764893E-01
b14 =	-1.3077234673E-04	a14 =	2.5258064287E-01
b05 =	-4.5104065313E-07	a05 =	-6.0555336300E-02
b51 =	5.8688819810E-17	a51 =	5.6130775062E-03
b42 =	-1.1001386949E-06	a42 =	5.2244484345E-01
b33 =	-1.0779522298E-06	a33 =	1.9765234785E+00
b24 =	-1.5196770287E-05	a24 =	1.3669939443E+00
b15 =	3.2744873891E-07	a15 =	-1.0318415241E-01
b06 =	-3.9790164297E-08	a06 =	-5.0533800754E-04
b52 =	7.8506286572E-16	a52 =	7.5084466289%-02

b43	7.6923051792E-16	a43	1.4865470557E-01
b34	1.1274391494E-16	a34	5.9769740165E-02
b25	-7.1735624312E-19	a25	-1.3335479815E-02
b16	3.3405178718E-19	a16	4.3268979607E-04
		a07	-3.2329508492E-05
		a44	1.0380388849E-13
		a35	1.0155064007E-13
		a26	-1.9818313447E-15
		a17	2.7141707709E-16

VI. CONCLUSIONS

A new method of designing 2-D IIR digital filters, with nonuniform samples of the desired frequency response, has been presented. The method is based on the use of an interpolating continued fraction which has a number of desirable properties such as

1) recursive computation of the design parameters
2) permanence
3) reality of the resulting transfer function.

The method, however does not quarantee the filter's stability. Suggestions for future work could include:

1) Extension to higher dimensions
2) Determination of continued fractions which not only approximate but also determine stability
3) Application of the 2-D continued fraction to system reduction problems
4) Investigation of the connections among the proposed continued fraction and 2-D moments and 2-D orthogonal expansions.

REFERENCES

1. L.R. Rabiner and B.Gold, "Theory and Applications of Digital Signal Processing", Prentice-Hall, 1975.
2. T.W. Parks and C.S. Burrus, "Digital Filter Design", Wiley, 1987.
3. J.E. Diamessis and C.W. Therrien, "Design of 1-D and 2-D FIR Filters with Nonuniform Frequency Samples", IEEE JCASSP Proc., April 1987.
4. J.E. Diamessis and C.W. Therrien, "2-D Interpolation, Matrix Factorization and Applications to Digital Signal Processing", IEEE ISCAS Proc., June 1988.
5. W.J. Rozwod, C.W. Therrien and J.S. Lim, "A Novel Method for Nonuniform Frequency Sampling Design of 2-D 2-D FIR Filters. IEEE ISCAS Proc., June 1988.
6. J.E. Diamessis and G.G. Potamianos, "Modeling Unequally Spaced 2-D Discrete Signals by Rational Functions", IEEE ISCAS Proc., May 1989.
7. J.E. Diamessis and G.G. Potamianos, "A Novel Method for designing IIR Filters with Nonuniform Samples", Submitted to Princeton University Conference on "Information Sciences and Systems", 1990.
8. J.S. Lim, "Two-dimensional Signal and Image Processing", Prentice-Hall 1990.

APPENDIX: Proof of Theorem 1.

1. "$\alpha'_{k\ell} = \alpha^*_{k\ell}$."

   For $|w_k|=1$, $|z|=1$ we have: $w^*_k = w^{-1}_k$, $z^* = z^{-1}$

   and

   $$G(w_k^{-1}, z_\ell^{-1}) = G(w_k, z_\ell) = G^*(w_k, z_\ell)$$

   (due to the structure of the CF(5)).
   Using these facts, the conclusion follows by direct computation of $\alpha_{k\ell}$ and $\alpha'_{k\ell}$.

2. " can be computed recursively from the data".

   For each w_k $(k=0,1,\ldots,m-1)$ we compute the values of $A_\ell(z)$ at $z=z_j$, $j=0,1,\ldots,n-1$. Then, knowing the values of $A_\ell(z)$ we let $z=z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_{n-1}, z_{n-1}^{-1}$ in (9) and obtain the values of $a_{\ell 1}, a_{\ell 2}, \ldots$ This is a recursive computation. Note that the values of $\alpha'_{k\ell}$ need not be computed, since they are equal to $\alpha^*_{k\ell}$.

3. "The $\alpha_{k\ell}$ are permanent".

   Each "new" coefficient $\alpha_{k\ell}$ does not depend on the previous coefficients (see steps in the computation).

4. "G(w,z) has real coefficients".

   It follows from the fact that $G_1(w,z)$ and $G_2(w,z)$ have conjugate complex coefficients, $\alpha'_k = \alpha^*_k$ (and we form their sum).

5. "Reduction of the 2-D CF to rational function form".

   This is accomplished in two steps. First we invert (to rational function) the 1-D ICF $A_\ell(z)$, $=0,1,\ldots$ using the fundumental recurrence relations of CF (or some other algorithm) and then we invert the 2-D ICF using a similar procedure.