# Allen's hourglass: probabilistic treatment of interval relations

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## Abstract

*The paper is motivated by the need to handle robustly the uncertainty of temporal intervals, e.g. as it occurs in automated event detection in video streams. The paper introduces a two-dimensional mapping of Allen's relations, based on orthogonal characteristics of interval relations, namely relative position and relative size. The hourglass-shaped mapping also represents limit cases that correspond to durationless intervals. Based on this mapping, we define two sets of primitive interval relations in terms of the relative positioning and relative size of intervals. These primitives are then used to derive a probabilistic set of Allen's relations. A number of example cases are presented to illustrate how the proposed approach can improve the robustness of interval relations.*

## 1. Introduction

Allen's interval calculus [1] has been used extensively for temporal reasoning [2], human mental modeling [6] and temporal databases [4]. In the context of automated information extraction from multimedia content [10], Allen's relations provide a mechanism for inferring higher level knowledge and/or querying over detected events. However, the rigid nature of qualitative relations does not encourage their direct use, since they do not suit the inexact nature of the information extraction algorithms [5]. In particular, as pointed out in [7], qualitative interval relations may not be robust enough to small temporal perturbations.

Addressing this issue, one may consider extending Allen's relations with uncertainty values. In [8], probability temporal interval networks are introduced, where nodes are intervals and edges are probabilistic relations. In [9] probabilities are attached to intervals, while in [3] a fuzzy extension of Allen's algebra has been proposed. In all of these approaches, the use of uncertainty is mostly motivated by the need to impose soft constraints on temporal relations, while the focus is on providing the algebra and algorithms to deal with uncertainty values, once these are available.

A somehow different issue, which is the focus of this article, is how to assign uncertainty values to Allen's relations, so as to compensate for the inherent uncertainty of the automated event detection process. Once uncertainty values are assigned to interval relations, they can be used, for example, as input to probability temporal interval networks to be checked for consistency [8] and to provide a reliable confidence index with respect to a database query [4]. We believe that this extension can improve significantly the robustness and the applicability of Allen's relations.

Assigning probabilities to interval relations, one has to account for symmetries between relations. Such symmetries have been studied in terms of reorientation and transposition transforms (see [6] and references therein). Nevertheless, to our knowledge, the relative size of intervals has not been used as a primary axis of symmetry. As we show here, it turns out the relative size, together with relative duration, create a two-dimensional space, which greatly facilitates the understanding of the relations and the assignment of probabilities.

Overall, this article has three main contributions. First, it sheds light to the geometry of Allen's relations by mapping them onto a two-dimensional hourglass configuration, where the horizontal axis corresponds to relative time and the vertical one to relative size of intervals. Along its vertical limits, the mapping also allows to visualize cases where either the target or the reference interval is durationless.

Second, for each dimension, a probability set of ordered primitive relations is defined, namely (before–concurrent with–after) and (smaller–equisized–larger). A sound way to determine the probabilities of these relations, as functions of the relative size and relative position of the intervals, is detailed.

Last, a probabilistic set of Allen's relations is presented and a way to evaluate the probabilities using the primitive relations, is defined.

The rest of the article is structured as follows. Section 2 explains the shortcomings of Allen's relations that have motivated our work. Section 3 introduces the two-dimensional

hourglass configuration of Allen's relations. Section 4 describes the primitive relation sets and their association to the relative size and duration of the intervals, whereas Section 5 presents the probabilistic extension of Allen's relations. Finally, Section 6 shows the utility of the defined extension for a number of cases and Section 7 summarizes the results and highlights open issues.

## 2. Motivation

**Allen's relations** An interval is defined as a set of real numbers with the property that any number that lies between any two numbers in the set is also included in the set. Allen (1983) introduced a calculus for representing the temporal relation of events delimited by time intervals. In particular, he defined 13 jointly exhaustive and pairwise disjoint (JEPD) qualitative relations between intervals. These are summarized in Table 1. To give an example, if X is a time

| name | definition | example |
|---|---|---|
| Before (b) | $X_b < X_e < Y_b < Y_e$ | |
| Meets (m) | $X_b < X_e = Y_b < Y_e$ | |
| Overlaps (o) | $X_b < Y_b < X_e < Y_e$ | |
| Starts (s) | $Y_b = X_b < X_e < Y_e$ | |
| During (d) | $Y_b < X_b < X_e < Y_e$ | |
| Finishes (f) | $Y_b < X_b < Y_e = X_e$ | |
| Equals (=) | $X_b = Y_b < Y_e = X_e$ | |
| Finishes-i (fi) | $X_b < Y_b < Y_e = X_e$ | |
| During-i (di) | $X_b < Y_b < Y_e < X_e$ | |
| Starts-i (fi) | $X_b = Y_b < Y_e < X_e$ | |
| Overlaps-i (oi) | $Y_b < X_b < Y_e < X_e$ | |
| Meets-i (mi) | $Y_b < Y_e = X_b < X_e$ | |
| After (a) | $Y_b < Y_e < X_b < X_e$ | |

**Table 1. Allen's qualitative relations between two intervals $X = [X_b, X_e]$ and $Y = [Y_b, Y_e]$.**

interval with boundaries $X = [10\text{sec}, 12\text{sec}]$ and Y is a time interval with boundaries $Y = [2\text{sec}, 40\text{sec}]$, then the (only) relation holding is: (X During Y). Although Allen's relations are complete, they may not fit very well to temporal segments that result from automated analysis, which are inherently uncertain. In what follows, we discuss two particular such shortcomings: lack of robustness and inadequacy.

**Lack of Robustness** Automated analysis results are inherently uncertain, due to the limited accuracy of the recognition algorithms. One source of uncertainty is the inaccuracy of segment boundaries. It is important that small differences of segmentation results should not result in significant differences in their inferred relations. This can happen with Allen's qualitative relations, for two reasons. First, several relations (such as Meets) require equality of time points, something which can only hold approximately in real-word extracted time intervals. Second, negligible changes to one of the end-points, with respect to the size of the time interval may result in a different qualitative relation.


**Figure 1. Allen's relations lack of robustness.**

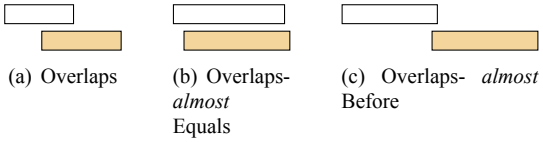



**Figure 2. Allen's relations inadequacy.**

To see that, let us examine the case of two intervals of approximately the same size. More formally, let a time duration $\epsilon$, such that $\epsilon < \alpha|X_i|, \forall x_i$ and $\alpha \in (0, 1)$ is a real number that can be chosen to be arbitrary small. Using $\epsilon$, we define a uniform random variable $U \sim \mathcal{U}(-\epsilon, \epsilon)$. Now let a referred time interval $X = [X_b, X_e]$ and let $Y = [X_b + u_1, X_e + u_2]$ where $u_1$ and $u_2$ are generated by $U$. By definition, interval Y is *almost* equal to X but not necessarily exactly equal. In fact, as depicted in Figure 1, several of Allen's relations may hold between X and Y, depending on $u_1$ and $u_2$. Consequently, the information that a particular relation holds between two intervals looses its significance, since most other could characterize approximately the same situation equally well.

**Inadequacy** Another issue with Allen's relations is their inadequacy in discriminating between very different situations occurring between intervals. This failure is a consequence of the fact that no quantitative information is taken into account. In particular, there are cases where a a different relation would result if the end points were slightly perturbed. Figure 2 depicts this failure in the case of three quite different situations, all being qualified as Overlaps.

## 3. A two-dimensional interval relation space

The idea that the 13 relations of Allen have some geometry is implicitly conveyed by the names of the relations: most of them come in pairs, where one is considered the

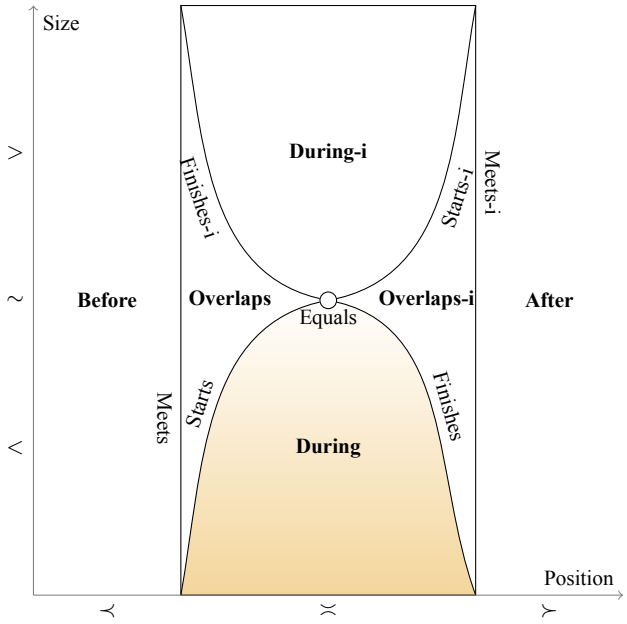



**Figure 3. Two-dimensional hourglass configuration of Allen's relations**

inverse of the other (e.g. Starts vs Starts-i). It is instructive to examine how the "inverse" attribute modifies each relation. In most cases, it implies reverting the direction of time. This is the case for the Before–After, Meets–Meets-i and Overlaps–Overlaps-i pairs.

Interestingly, the pair During–During-i cannot be obtained by the same principle: if X happens During Y then it would still happen During Y if the time axis was reversed. To obtain During-i we have to actually change the relative size of the intervals, i.e. letting X being larger than Y, instead of being smaller. Furthermore, a careful examination of the Starts–Starts-i and Finishes–Finishes-i pairs reveals that the inverse characteristic refers to both relative time position and relative size.

**The Hourglass Configuration** This observation led us to explore a two-dimensional configuration of Allen's relations, where one dimension corresponds to the relative position of the intervals and the other to their relative size. Figure 3 shows how these relations may be depicted in a two-dimensional configuration that resembles an hourglass. The horizontal axis corresponds here to the relative position of the intervals, while the vertical one to their relative size. The coordinate axes also introduce some symbols that correspond roughly to the probabilistic primitive relations that will be defined in Section 4 ($\prec$: **before**, $\asymp$: **concurrent with**, $\succ$: **after**, $>$: **larger**, $\sim$: **equisized** and $<$: **smaller**).

The hourglass configuration has some nice properties. First, by letting Y be a fixed interval and X an interval that progressively moves forward in time, the hourglass enables us to quickly visualize the relations it will go through, provided that we chose a particular relative size of intervals. For example, when X has the same size as Y, these are: Before, Meets, Overlaps, Equals, Overlaps-i, Meets-i and After.

Moreover, the hourglass allows to distinguish between relations that are defined solely by inequalities from those that are defined also by equalities. The *regions* inside and outside the hourglass correspond to the six relations defined by inequalities: Before, Overlaps, During, During-i, Overlaps-i and After. The *lines* defining these regions correspond to the six relations that include an equality in their definition: Meets, Starts, Finishes, Finishes-i, Starts-i and Meets-i. Last, the central *point* of the hourglass corresponds to the only relation defined by two equalities, namely the Equals relation.

Another useful property of the hourglass configuration is that it reveals the two distinct axes of symmetry of the relations: "folding" the hourglass along the horizontal axis and around the center (relative position symmetry) makes, for example, the Before and After regions coincide, whereas "folding" it along the vertical axis (relative size symmetry) makes the During and During-i regions coincide.

The hourglass also distinguishes between relations with respect to of the relative size of the intervals: the relations appearing at its lower part (Starts, During and Finishes) are applicable only if the first interval is smaller than the second. whereas the relations appearing only at its upper part (Finishes-i, During-i and Starts-i) are applicable only if the first interval is larger than the second, All other relations are applicable regardless of the relative size of the intervals.

**The Limits of the Hourglass** The hourglass configuration allows us also to visualize the relations applicable in three limit cases, with respect to the relative size of the intervals. Namely, the lower line (bottom) of the hourglass corresponds to the case where the first interval is infinitely smaller than the second, i.e. it reduces to a durationless interval. Notice that, as one should expect, the Overlaps and Overlaps-i relations do not apply here: at the limit, a time point cannot overlap with an interval. Still on the lower line, the Meets–Starts and Finishes–Meets-i pairs become indistinguishable when the first interval is infinitely smaller than the second. As a result, they are displayed as single points at the lower left and right corner of the hourglass.

Similar conclusions can be drawn by looking at the upper line (top) of the hourglass, where the first interval is infinitely larger than the second, i.e. the second interval becomes durationless. Here, the Overlaps and Overlaps-i relations also do not apply, while the Meets–Finishes-i and the Starts-i–Meets-i relations converge to the upper left and

right corners of the hourglass.

A third limit case worth mentioning is when the intervals have the same size. In this case, one can easily verify that the Starts, Starts-i, Finishes and Finishes-i relations coincide with the Equals relation.

## 4. Primitive probabilistic relations

Motivated by the two-dimensional configuration of Allen's relations, and in order to extend the relations with probabilities, we follow a two-step approach. At the first step, discussed in this section, we define a number of *primitive* interval relations for each of the two dimensions. In order to distinguish them from Allen's original relations, we will denote the new ones with bold letters. These relations are not binary (hold/do not hold) but quantitative, i.e. they hold up to a certain degree, between zero and one. We shall discuss how this degree may be given a probabilistic interpretation.

At a second step, deferred to Section 5, we use these primitives as a basis for a probabilistic extension of Allen's relations defined by inequalities.

Note that, in all definitions, we assume intervals to have strictly positive size.

### 4.1. Primitives for relative position

**The Concurrent relation** Let us first define the **concurrent with** primitive, which we will mathematically denote as $\asymp$. To do this, let us see how Allen's During is defined and see how we may extend it and attach a probability measure. Allen's During holds iff X is entirely into Y. As shown in Figure 1, even if a small part of X is not into Y then this relation will not hold.

In case the relation holds, we may say that "the probability of X being concurrent with Y" equals 1. However, we may also relax the definition so that when the part of X not in Y is very small, then the **concurrent with** relation *almost* holds. To that end, we adopt the following view:

> Let a point $t$ chosen randomly in X. What is the probability that $t$ is also in Y?

In other words, we measure the probability $P(\mathrm{X} \asymp \mathrm{Y})$ by considering $P(t \in \mathrm{Y} | t \in \mathrm{X})$.

In order to define a symmetric relation, instead of choosing $t$ from X in all cases, we will be choosing $t$ randomly from the smallest interval, and thus define $P(\mathrm{X} \asymp \mathrm{Y})$ as

$$P(\mathrm{X} \asymp \mathrm{Y}) \stackrel{def}{=} \begin{cases} P(t \in \mathrm{Y} | t \in \mathrm{X}) & |\mathrm{X}| \leq |\mathrm{Y}|, \\ P(t \in \mathrm{X} | t \in \mathrm{Y}) & |\mathrm{X}| > |\mathrm{Y}| \end{cases} \quad (1)$$

where $|\cdot|$ denotes the *size* of the interval:

$$|\mathrm{X}| = \mathrm{X}_e - \mathrm{X}_b$$

Note that $P(\mathrm{X} \asymp \mathrm{Y}) = P(\mathrm{Y} \asymp \mathrm{X})$. Moreover, we will make the assumption that there is no reason to choose one time point over another. In this case, Eq. (1) simplifies to

$$P(\mathrm{X} \asymp \mathrm{Y}) = \begin{cases} \frac{|\mathrm{X} \cap \mathrm{Y}|}{|\mathrm{X}|} & |\mathrm{X}| \leq |\mathrm{Y}|, \\ \frac{|\mathrm{X} \cap \mathrm{Y}|}{|\mathrm{Y}|} & |\mathrm{X}| > |\mathrm{Y}| \end{cases}$$

When this ratio equals 1, then the smallest interval is entirely into the larger one. Smaller probability values indicate that some part of the smaller interval is not into the larger one. Last, zero probability will indicate that no part of one interval is within the other.

**The Before and After relations** The **before** and **after** relations will be denoted as $\prec$ and $\succ$ respectively. When *some part* of X is not in Y then X is either before or after Y. We will first deal with the case where X is not larger than Y, to ensure that all parts of X not in Y will be either *only* before Y or *only* after Y but not both.

For this definition, we need to introduce some notation. Namely, the left complement of the interval $\mathrm{Y} = [\mathrm{Y}_b, \mathrm{Y}_e]$, i.e. the interval that covers the entire time axis until $\mathrm{Y}_b$, is denoted as $\overleftarrow{\mathrm{Y}}$:

$$\overleftarrow{\mathrm{Y}} = (-\infty, \mathrm{Y}_b)$$

Similarly, the right complement of Y, i.e. the interval that covers the entire time starting from $\mathrm{Y}_e$, is denoted as $\overrightarrow{\mathrm{Y}}$:

$$\overrightarrow{\mathrm{Y}} = (\mathrm{Y}_e, +\infty)$$

Now, following the same arguments as with the **concurrent with** relation, the probability of X being **before** Y is defined as the probability of choosing a random point in X that lies before Y, i.e. in $\overleftarrow{\mathrm{Y}}$. Formally,

$$\begin{aligned} \forall |\mathrm{X}| \leq |\mathrm{Y}| : \quad \Pr(\mathrm{X} \prec \mathrm{Y}) &\stackrel{def}{=} P(t \in \overleftarrow{\mathrm{Y}} | t \in \mathrm{X}) \\ &= \frac{|\mathrm{X} \cap \overleftarrow{\mathrm{Y}}|}{|\mathrm{X}|} \end{aligned} \quad (2)$$

Along the same line of argument, the probability of X being **after** Y is defined as:

$$\begin{aligned} \forall |\mathrm{X}| \leq |\mathrm{Y}| : \quad \Pr(\mathrm{X} \succ \mathrm{Y}) &\stackrel{def}{=} P(t \in \overrightarrow{\mathrm{Y}} | t \in \mathrm{X}) \\ &= \frac{|\mathrm{X} \cap \overrightarrow{\mathrm{Y}}|}{|\mathrm{X}|} \end{aligned} \quad (3)$$

Now, when $|\mathrm{X}| > |\mathrm{Y}|$, the same definitions hold but with

the $X$ and $Y$ arguments inverted. Namely:

$$\forall |\mathrm{X}| > |\mathrm{Y}| : \quad \Pr(\mathrm{X} \prec \mathrm{Y}) \stackrel{def}{=} P(t \in \overrightarrow{\mathrm{X}} | t \in \mathrm{Y}) = \frac{|\mathrm{Y} \cap \overrightarrow{\mathrm{X}}|}{|\mathrm{Y}|} \tag{4}$$

$$\forall |\mathrm{X}| > |\mathrm{Y}| : \quad \Pr(\mathrm{X} \succ \mathrm{Y}) \stackrel{def}{=} P(t \in \overleftarrow{\mathrm{X}} | t \in \mathrm{Y}) = \frac{|\mathrm{Y} \cap \overleftarrow{\mathrm{X}}|}{|\mathrm{Y}|} \tag{5}$$

One may easily verify that the **before** and **after** relations are inversely related, i.e

$$\forall \mathrm{X}, \mathrm{Y} : \quad \Pr(\mathrm{X} \prec \mathrm{Y}) = \Pr(\mathrm{Y} \succ \mathrm{X})$$

**Probability base with respect to relative position** Altogether, the three relative position primitives introduced above (**before**, **concurrent with** and **after**) are complementary: the more one holds, the less the others do. In particular, the primitives form a complete base, in the sense that, for any two intervals X and Y:

$$\Pr(\mathrm{X} \prec \mathrm{Y}) + \Pr(\mathrm{X} \asymp \mathrm{Y}) + \Pr(\mathrm{X} \succ \mathrm{Y}) = 1$$

One may easily verify this, when $|\mathrm{X}| \leq |\mathrm{Y}|$:

$$\Pr(\mathrm{X} \prec \mathrm{Y}) + \Pr(\mathrm{X} \asymp \mathrm{Y}) + \Pr(\mathrm{X} \succ \mathrm{Y}) = \frac{|\mathrm{X} \cap \mathrm{Y}| + |\mathrm{X} \cap \overleftarrow{\mathrm{Y}}| + |\mathrm{X} \cap \overrightarrow{\mathrm{Y}}|}{|\mathrm{X}|} = \frac{|\mathrm{X} \cap (\mathrm{Y} \cup \overleftarrow{\mathrm{Y}} \cup \overrightarrow{\mathrm{Y}})|}{|\mathrm{X}|} = \frac{|\mathrm{X}|}{|\mathrm{X}|} = 1$$

and similarly for the $|\mathrm{X}| > |\mathrm{Y}|$ case.

## 4.2. Primitives for relative size

We now discuss the second dimension of interval relations, namely the relation between interval sizes. In what follows we define three relative duration primitives that correspond roughly to the following cases:

- both intervals have the same size
- the first interval is smaller than the second
- the first interval is larger than the second

Consider the measured ratio

$$\hat{s} = \frac{|\mathrm{X}| - |\mathrm{Y}|}{|\mathrm{X}| + |\mathrm{Y}|}$$

that takes values from $(-1, +1)$ where the upper, respectively the lower, limit correspond to the case $X$ has "infinitely" greater, respectively smaller, size than $Y$. Assuming that the boundaries of intervals are uncertain, we may consider that $\hat{s}$ is the mean value of a random variable $\mathcal{S}$, following the normal distribution $\mathcal{S} \sim \mathcal{N}(\hat{s}, \sigma)$.

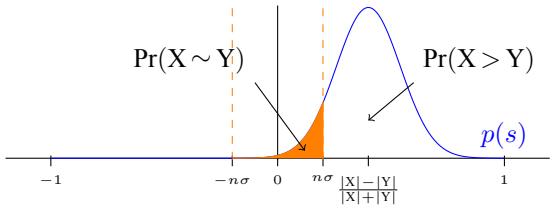



**Figure 4. The equisized and larger probabilities as areas of $\mathcal{N}(s, \sigma)$, delimited by $-n \cdot \sigma$ and $n \cdot \sigma$. Here, smaller is negligible and not explicitly depicted.**

We may then define the probability of X being **larger** (in short $>$), **equisized** (in short $\sim$) and **smaller** (in short $<$) than Y, as the cumulative probability of $S$ taking negative values, close to zero or positive values respectively. Figure 4 illustrates these definitions. Formally, by letting $n$ be a parameter $n \in \Re^+$, this is expressed as:

$$\Pr(\mathrm{X} < \mathrm{Y}) = \Phi(-n \cdot \sigma | \hat{s}, \sigma) \tag{6}$$

$$\Pr(\mathrm{X} \sim \mathrm{Y}) = \Phi(n \cdot \sigma | \hat{s}, \sigma) - \Phi(-n \cdot \sigma | \hat{s}, \sigma) \tag{7}$$

$$\Pr(\mathrm{X} > \mathrm{Y}) = 1 - \Phi(n \cdot \sigma | \hat{s}, \sigma) \tag{8}$$

Furthermore, by making use of the complementary error function:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

the above definitions reduce to

$$\Pr(\mathrm{X} < \mathrm{Y}) = \frac{1}{2} \mathrm{erfc}\left(n + \frac{\hat{s}}{\sigma}\right)$$

$$\Pr(\mathrm{X} \sim \mathrm{Y}) = 1 - \frac{1}{2}\left(\mathrm{erfc}\left(n - \frac{\hat{s}}{\sigma}\right) + \mathrm{erfc}\left(n + \frac{\hat{s}}{\sigma}\right)\right)$$

$$\Pr(\mathrm{X} > \mathrm{Y}) = \frac{1}{2} \mathrm{erfc}\left(n - \frac{\hat{s}}{\sigma}\right)$$

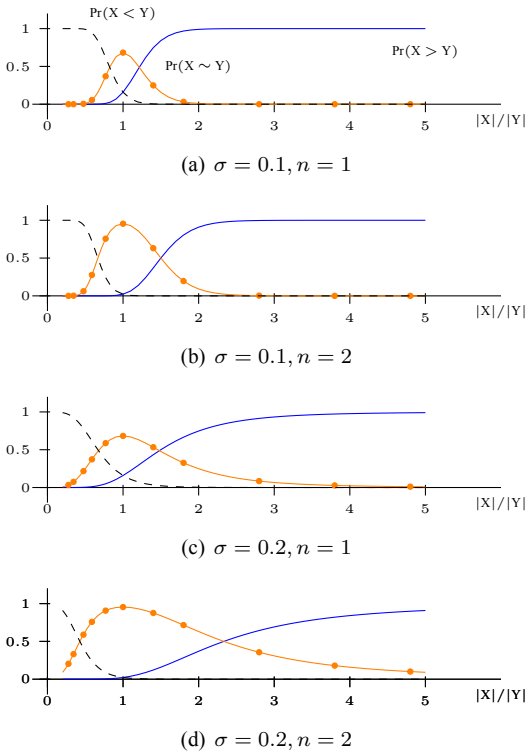
**Probability base for relative duration** Altogether, the **smaller**, **equisized** and **larger** primitives are complementary. One may easily verify that:

$$\Pr(\mathrm{X} < \mathrm{Y}) + \Pr(\mathrm{X} \sim \mathrm{Y}) + \Pr(\mathrm{X} > \mathrm{Y}) = 1$$

i.e. they form a complete probability base.

**Dependence on the $\sigma$ and $n$ parameters** Figure 5 illustrates how relative size probability primitives are affected when varying $\sigma$ and $n$.

These parameters allow us to model our uncertainty towards the measurement and our tolerance in considering two




(a) $\sigma = 0.1, n = 1$

(b) $\sigma = 0.1, n = 2$

(c) $\sigma = 0.2, n = 1$

(d) $\sigma = 0.2, n = 2$

**Figure 5. Relative size primitives as a function of the $\sigma$ and $n$ parameters.**

intervals to be **equisized**. Namely, $\sigma$ quantifies our uncertainty regarding the relative size measurement: small values of $\sigma$, e.g. $\sigma \ll 0.1$ should be used when there is a significant confidence regarding the measurement, whereas large values when measurement confidence is low.

On the other hand, $n$ regulates our tolerance in favor of the **equisized** relation. To see the consequences of choosing a particular $n$, let us examine the case when the measured relative size ratio is zero ($s = 0$). This is the case where $\Pr(\mathrm{X} \sim \mathrm{Y})$ is maximized. The maximum value obtained is not 1, but equals the area of the normal distribution $\mathcal{N}(0, \sigma)$ within the confidence interval $(-n\sigma, +n\sigma)$:

$$\Pr(\mathrm{X} \sim \mathrm{Y})_{\hat{s}=0} = \mathrm{erf}(\frac{n}{\sqrt{2}})$$

For example, when $n = 1$, respectively $n = 2$ the maximum value of **equisized** obtainable is $0.68$, respectively $0.95$.

## 5. Probabilistic extension of Allen's relations

In the previous section, we defined six primitive relations: **before**, **concurrent with**, **after** and **smaller**, **equisized**, **larger**. One may use them directly to describe the relation between intervals. However, our main motivation has been to use them as a basis upon which to build a probabilistic extension of Allen's relations. Extending Allen's relations with probabilities means that two intervals may be related by more than one relations (e.g. During and Overlaps) with specific probability degrees and that these degrees should add to 1.

We now describe this extension. To keep notation simple, we will use the same names and symbols as Allen's. The difference will be evident by the probability symbol. To give an example, the fact that "the probability that X and Y are related by Allen's During-i relation (di) is 0.7" will be denoted as $\Pr(\mathrm{X\,di\,Y}) = 0.7$.

The reader can easily verify that the proposed extension will form a probability set, in the sense that:

$$\begin{aligned}&\forall \mathrm{X}, \mathrm{Y}: \\ &\quad \Pr(\mathrm{X\,b\,Y}) + \Pr(\mathrm{X\,o\,Y}) + \Pr(\mathrm{X\,d\,Y}) \\ &\quad + \Pr(\mathrm{X\,di\,Y}) + \Pr(\mathrm{X\,oi\,Y}) + \Pr(\mathrm{X\,a\,Y}) = 1\end{aligned}$$

### 5.1. The before and after relations

From Figure 3 one may observe that Allen's Before and After relations are independent of the relative size dimension. Therefore, their probabilistic extension should also be independent of the relative size of the intervals. Namely, we may use directly the relative position primitives, as defined in Eq. (2)-(5), and define Before and After simply as

$$\begin{aligned}\Pr(\mathrm{X\,b\,Y}) &\stackrel{def}{=} \Pr(\mathrm{X} \prec \mathrm{Y}) \\ \Pr(\mathrm{X\,a\,Y}) &\stackrel{def}{=} \Pr(\mathrm{X} \succ \mathrm{Y})\end{aligned}$$

Note that we have used the symbols b and a for Allen's relations instead of the commonly used $<$ and $>$ which are used here for the relative size primitives.

### 5.2. The during and during-i relations

The definition of these relations is facilitated by three limits along the relative duration axis of the hourglass, i.e. when the first interval is (a) infinitely smaller, (b) equisized and (c) infinitely larger than the second.

At the infinitely smaller limit, the probability of During-i will always be zero, whereas the probability of During is complementary to the probability of being before or after, since no other relation is possible. Therefore the probability of During is measured here directly by the **concurrent with** primitive as defined in Eq. (1):

$$\begin{aligned}\Pr(\mathrm{X} < \mathrm{Y}) \to 1: \quad &\Pr(\mathrm{X\,d\,Y}) = \Pr(\mathrm{X} \asymp \mathrm{Y}) \\ &\Pr(\mathrm{X\,di\,Y}) = 0\end{aligned}$$

$$\lim_{\Pr(X<Y)\to 1} \Pr(X\,d\,Y) = \Pr(X \asymp Y)$$
$$\lim_{\Pr(X<Y)\to 1} \Pr(X\,di\,Y) = 0$$

Following similar arguments, for the infinitely larger limit, the following constraints should be respected:

$$\Pr(X>Y)\to 1: \quad \Pr(X\,di\,Y) = \Pr(X \asymp Y)$$
$$\Pr(X\,d\,Y) = 0$$

Finally, at the equisized limit, both probabilities should go to zero:

$$\Pr(X\sim Y)\to 1: \quad \Pr(X\,d\,Y) = \Pr(X\,di\,Y) = 0$$

Taking into account that both $\Pr(X<Y)$ and $\Pr(X>Y)$ go to zero when $\Pr(X\sim Y)$ goes to one (see Eq.(6)-(8)), all the above constraints are satisfied by the following definitions:

$$\Pr(X\,d\,Y) \stackrel{def}{=} \Pr(X<Y)\Pr(X \asymp Y)$$

and

$$\Pr(X\,di\,Y) \stackrel{def}{=} \Pr(X>Y)\Pr(X \asymp Y)$$

### 5.3. The overlaps and overlaps-i relations

Similarly to the During and During-i cases, a number of constraints may be derived for the three limit cases. Namely:

$$\Pr(X<Y)\to 1 : \Pr(X\,o\,Y) = \Pr(X\,oi\,Y) = 0$$
$$\Pr(X\sim Y)\to 1 : \Pr(X\,o\,Y) + \Pr(X\,oi\,Y) = \Pr(X \asymp Y)$$
$$\Pr(X>Y)\to 1 : \Pr(X\,o\,Y) = \Pr(X\,oi\,Y) = 0$$

These can be verified easily by looking at the hourglass configuration in Figure 3.

In particular, the second constraint here implies that when two intervals are equisized and they overlap, whether left or right, the overall extent of the overlap is given by the ratio of their intersection to their total size. Moreover, one may notice that the probability of Overlaps (respectively the Overlaps-i) relation is correlated to the probability of Before (respectively After) relation and that Overlaps and Overlaps-i can not be simultaneously non-zero. Therefore, one may add the following constraints:

$$\Pr(X\sim Y)\to 1, \Pr(X\succ Y) = 0 :$$
$$\Pr(X\,o\,Y) = \Pr(X \asymp Y)$$
$$\Pr(X\,oi\,Y) = 0$$

and

$$\Pr(X\sim Y)\to 1, \Pr(X\succ Y) > 0 :$$
$$\Pr(X\,o\,Y) = 0$$
$$\Pr(X\,oi\,Y) = \Pr(X \asymp Y)$$

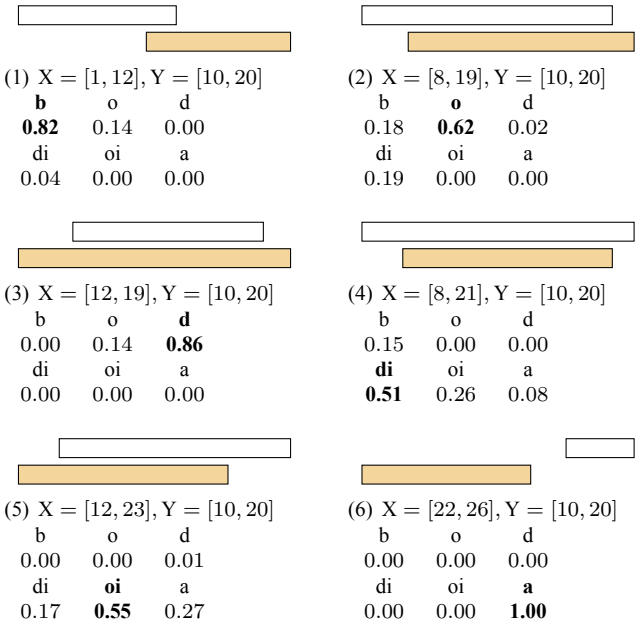



**Figure 6. Expressiveness and Robustness of probabilistic primitives for $\sigma = 0.10, r = 0.10$**

Following similar arguments to Section 5.2, these relations are defined as:

$$\Pr(X\,o\,Y) \stackrel{def}{=} \begin{cases} \Pr(X\sim Y)\Pr(X \asymp Y) & \Pr(X\succ Y) = 0 \\ 0 & \Pr(X\succ Y) = 0 \end{cases}$$

and

$$\Pr(X\,oi\,Y) \stackrel{def}{=} \begin{cases} \Pr(X\sim Y)\Pr(X \asymp Y) & \Pr(X\succ Y) > 0 \\ 0 & \Pr(X\succ Y) = 0 \end{cases}.$$

## 6. Examples

To illustrate how the proposed approach improves the robustness and expressiveness of interval relations, we present here its application to examples similar to those described in Section 2. In particular, Figure 6 illustrates cases with different (qualitative) interval relations, namely Overlaps, During, Overlaps-i and During-i and After. Using the probabilistic extension proposed here, probabilities are obtained for several relations in each case. The most probable relation is highlighted with bold letters.

In almost all cases, the most probable relation coincides with the qualitative one, i.e. the one that would have been obtained by directly applying Allen's definitions. However, probabilities provide additional expressiveness, since their values depend on the extent to which each relation holds. For instance, the Overlaps relation in case (2) has greater

probability than the Overlaps-i relation in case (5). This is because in case (5), the intervals overlap more than in case (5).

Furthermore, by applying the probabilistic extension proposed here, relations become more robust with respect to boundary uncertainties. In particular, as one may notice by looking in case (1), a direct application of Allen's definitions would qualify the relation between X and Y as Overlaps. However, this hides the fact that X is *almost before* Y and the fact that they overlap may be due to uncertain measurements. Using the probabilistic extension, the information regarding the intervals relation is richer: both Before and Overlaps have significant probabilities, while Before is the most probable one.

## 7. Conclusion and Open Issues

The two-dimensional mapping of Allen's interval relations onto parts of an hourglass configuration has been proven constructive in two ways. First, it allowed us to decompose interval relations into two basic orthogonal dimensions: relative position and relative size of intervals. This has motivated the definition of two primitive interval relation sets, whose probabilities are defined independently on one of the two dimensions. Second, it simplified the probabilistic extension of Allen's relations by revealing the symmetries of relations and shedding light upon the expected behavior at the durationless and equisized limits.

To our knowledge', this is the first attempt to assign probability values to interval relations, aiming to make reasoning with uncertain temporal relations more practical. Both probabilistic primitive relations and probabilistic Allen's relations can be used to quantitatively describe the relation between two intervals. However, the proposed set of probabilistic Allen's relations cannot account for relations defined by equalities (e.g. Allen's Meets relation). Therefore, an open issue is to extend the current approach and define a probability base for Allen's relations defined by both inequalities and equalities.

Finally, the framework defined in this paper may be used in various ways that deviate from the event detection scenario introduced in this article. For instance, in the context of constraint reasoning for planning and scheduling, one may be given directly the probabilities of Allen relations that hold between pairs of intervals. The goal in this case, would be to derive information about the relative size and position of the corresponding intervals. This is applicable to soft-deadline problems and scheduling tasks.

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