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A New Approach for Multilevel Threshold Selection

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This paper describes a new method for multilevel threshold selection of gray level images. The proposed method includes three main stages. First, a hill-clustering technique is applied to the image histogram in order to approximately determine the peak locations of the histogram. Then, the histogram segments between the peaks are approximated by rational functions using a linear minimax approximation algorithm. Finally, the application of the one-dimensional Golden search minimization algorithm gives the global minimum of each rational function, which corresponds to a multilevel threshold value. Experimental results for histograms with two or more peaks are presented. © 1994 Academic Press, Inc.

1. INTRODUCTION

Thresholding is one of the most powerful techniques for image segmentation. The application of the thresholding technique is based on the assumption that object and background pixels in a digital image can be distinguished by their gray-level values [1]. In some cases, such as text images, it is a priori known that the image contains only two principal gray tones. The histogram of such an image may be considered that it represents the distribution of the image brightness. Using the histogram form, it is possible to determine an optimal threshold value for segmenting the image into the two brightness regions. The result of this processing is an image with only two gray levels, which correspond to the background and objects. This approach is referred to as global (or bilevel) thresholding. Over the past years, several techniques have been proposed for automatic global threshold selection. For a survey of thresholding techniques, see [2].

Global thresholding methods can be applied only to some images, where a clear foreground–background relationship exists [3]. For the segmentation of more complex images, however, it is necessary to resort to multilevel threshold selection techniques. Generally, it is not a simple matter to determine multilevel threshold values. In multi-object images there are several difficulties for multilevel threshold selection that are associated with the gray-level distributions, small objects, and object overlapping. To overcome these difficulties, several techniques have been proposed. Baukharouba *et al.* [4] first determine a distribution rational function. Then, the multithreshold values are defined as the zeros of a curvature function derived from the distribution function. Many multithresholding approaches are based on edge matching and classification. The methods of Wang and Haralick [5], Hertz and Schafer [6], Kohler [3], and Spann and Wilson [7] belong to this category. These methods are applicable to images with good edges. Additionally, they are not based on histogram information for determination of the threshold values. As a first step, in all these methods, the pixels of the initial image are first classified as edge and nonedge pixels by using an edge extraction algorithm. Consequently, for the extraction of the best thresholds, computationally expensive recursive procedures are used. During each iteration, the threshold values are modified in order to satisfy some edge characteristics.

Spann and Wilson [7] propose a hybrid multithreshold selection method, which is based on statistical and spatial information. Specifically, the method is a combination of a quad-tree smoothing technique, a local centroid clustering algorithm, and a boundary estimation approach. This method is applicable under some conditions, such as requiring that the histogram consist of only Gaussian distributions.

Another category of multithreshold selection approaches involves methods that are only based on image histogram. The methods of Reddi *et al.* [8], Kapur *et al.* [9], and Carlotto [10] belong to this category. The method of Reddi *et al.* is very fast and it is a version extended to multithresholding of the global threshold method of Otsu [11], which, according to [2], is probably the most powerful method for global thresholding. The criterion used is the selection of the threshold so that the interclass variance between dark and bright regions is maximized. A second interesting and effective multithreshold approach is the method of Kapur *et al.*, which is based on the maximum entropy criterion. An interesting approach for multithresholding has been proposed by Carlotto. According to this approach, the determination of the threshold values is achieved by handling information derived from the changes of zero-crossings in the second deriva-

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tive. This method gives good results only for the histograms that satisfy the basic hypothesis that the histogram consist of only univariate normal distributions.

This paper proposes a new, histogram-based multilevel threshold selection method that results in multilevel threshold values. The proposed method consists of three main steps. In the first stage, an efficient hill-clustering technique is applied and the peak locations of the histogram are approximately determined [12]. This technique is iterative and converges if the number of peaks found is less than or equal to the desirable maximum number of peaks. The only input value needed is the number of histogram peaks, or equivalently, the number of objects that are possibly included in the image. In the second stage, the histogram segments between the peaks are approximated by real rational functions using a fast linear programming algorithm and the minimax criterion [13, 14]. This method results in real rational functions which optimally approximate the valley points of the histogram by sufficiently small minimax approximation errors. Therefore, with this procedure, the histogram segments between the peaks are satisfactorily fitted by real rational functions. Finally, the multilevel threshold values of the histogram are defined as the global minimum values of each rational function. To find these global optimal values, the one-dimensional Golden search algorithm is applied [15]. It is noted that the linear programming problem always has a global solution, and the approximation errors are minimax.

To demonstrate the proposed method, experimental results for histograms with two or more peaks are presented. In order to assess the performance of the algorithm, the results are compared with those given by the methods of Otsu [11], Reddi *et al.* [8], and Kapur *et al.* [9].

2. PEAK DETERMINATION BY THE HILL-CLUSTERING METHOD

Let us consider a gray-level image that contains objects and background. For this image each object corresponds to a hill in the gray-level histogram. The histogram of such an image is a nonnegative, real-valued function and contains peaks and valleys. Pixels that lie in the neighborhood of peaks are classified as object pixels, whereas valley pixels are characterized as unclassified because they do not belong positively to a specific object.

Let $\Re$ be the set of the positive integers representing the image gray levels, and $f(x, y)$ the image function which gives the gray-level value of the pixel with coordinates (x, y). The histogram $H(k)$ of this image is defined as

$$H(k) = \sum_{\forall f(x,y)=k} f(x, y), \quad k \in \Re. \tag{1}$$

The multilevel threshold selection can be considered as the problem of finding a set $T(l)$, $l = 1, \ldots, L$, of threshold values, in order that the original gray-level image be transformed to a new one with $L + 1$ levels. This can be done by finding threshold values that are located in valleys and between adjacent hills. More specifically, if $T(l)$, $l = 1, \ldots l$, are the threshold values with $T(1) < T(2) < \ldots < T(L)$, then the resultant image is defined as

$$F(x, y) = \begin{cases} 0, & \text{if } f(x, y) \leq T(1), \\ 1, & \text{if } T(1) \leq f(x, y) \leq T(2), \\ & \vdots \\ L, & \text{if } f(x, y) \geq T(L). \end{cases} \tag{2}$$

The proposed multilevel threshold selection approach consists of three main parts. The first part is based on the hill-clustering method proposed by Tsai and Chen [12]. According to this method, the locations of the histogram peaks can be approximately determined by an iterative procedure. In each iteration the number of gray levels (cells) is reduced by half. More specifically, this method consists of the following steps:

Step 1. Give the value of M_o, where M_o is the maximum number of desired histogram peaks. The iterative clustering procedure converges when the total number of determined peaks is less than or equal to M_o. Also, set $ITER = 0$.

Step 2. Set $ITER = ITER + 1$ and the number of cells equals to $N_{C,ITER} = N_{CO}/2^{ITER-1}$, where N_{CO} the number of cells of the initial histogram. The frequency of each cell f_i is defined as $g_{i,ITER} = 0.5(g_{i,ITER-1} + g_{i-1,ITER-1})$, $i = 1, 2, \ldots, N_{C,ITER}$.

Step 3. Establish the arrow directions. If d_i is the arrow direction at cell i, then

$$d_i = \begin{cases} +1, & \text{if } (f_{i-1} > f_{i+1}) \cap (f_{i-1} \geq f_i) \neq 0, \\ -1, & \text{if } (f_{i+1} > f_{i-1}) \cap (f_{i+1} \geq f_i) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In the case of tie (i.e., $f_{i-1} = f_{i+1}$), set $d_i = d_{i-1}$.

Step 4. Identify the peaks:

(a) if $(d_i = 0) \cap (d_{i-1} = -1) \cap (d_{i+1} = +1)$, then cell i is the peak of the hill; or

(b) if $(d_i = -1) \cap (d_{i+1} = +1)$, then a peak is also identified between cells i and $i + 1$. Repeat Step 4 to identify all peaks in the gray-level histogram.

Step 5. If the number of peaks identified in Step 4 is less than or equal to the desired number of peaks, then go to Step 6; otherwise, go to Step 2.

Step 6. Approximate the locations of the histogram peaks and terminate the procedure. Specifically, if cell i is a peak then its gray level is identified as the location of the peak.

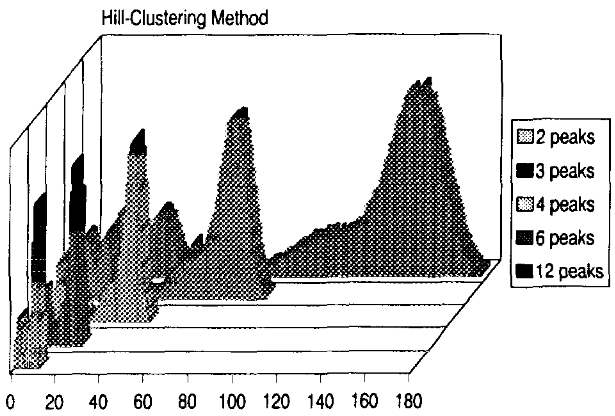

FIG. 1. An example of the hill-clustering method.

It is noted that the peak locations are only approximately determined. Also, the final number of peaks corresponds to the total number of hills and consequently to the total number of objects. Figure 1 shows the application of the hill-clustering method to a histogram with only two needed peaks.

3. MULTILEVEL THRESHOLDING PROBLEM FORMULATION

In the second part of the proposed method, the problem that must be solved is the optimum fitting of the histogram data between the peaks. This approximation must satisfy several characteristics such as small approximation errors, global convergence to the optimum solution, and low computational cost. To achieve these goals, we propose a new technique that is based on the linear rational approximation (LRA) algorithm [13, 14]. With this algorithm we can fit any set of data by a real rational function according to the minimax criterion. Additionally, the optimization technique that is used is the well-known linear programming technique and specifically the revised simplex algorithm [16]. The result of this procedure is the optimum approximation of histogram segments by real rational functions.

To clarify the approximation technique, let us consider that hill-clustering method gives a total number of histogram peaks equal to P, where $(W_n, G(W_n))$, $n = 1, \ldots, P$, are the coordinates of the peaks. This means that the histogram has $P - 1$ valleys, which lie between the peaks. For the nth valley let:

- K be the total number of gray levels included;
- $G(w_k)$, $k = 1, \ldots, K$, be the values of the histogram at the w_k gray level;
- w_k, $k = 1, \ldots, K$, be the gray-level values, with $W_n \leq w_k \leq W_{n+1}$.

For each valley, the LRA algorithm can fit the histogram points $(w_k, G(w_k))$, $k = 1, \ldots, K$, by a real rational function of the general form

$$R(w) = \frac{A(w)}{B(w)} = \frac{\sum_{m=0}^{N} a_m w^m}{1 + \sum_{m=1}^{M} b_m w^m}, \tag{4}$$

where a_m and b_m are the unknown coefficients, and N and M are integers that define the degree of the polynomials $A(w)$ and $B(w)$. This approximation problem is solved by a well-established linear programming approach based on the minimax criterion. Specifically, for every point $(w_k, G(w_k))$ the following objective criterion is defined:

$$G(w_k) - R(w_k) \rightarrow E_k, \tag{5}$$

Here the E_k are known quantities with small absolute values.

Also, for every k the variables ξ_k are defined as

$$\xi_k = (G(w_k) - E_k)B(w_k) - A(w_k), \tag{6}$$

which may also be written as

$$G(w_k) - E_k = R(w_k) + \xi_k / B(w_k). \tag{7}$$

According to the LRA method, the following approximation problem is formulated:

$$\begin{aligned} &\text{maximize } d' \\ &\text{subject to:} \\ &|(G(w_k) - E_k)B(w_k) - A(w_k)| \leq \xi \\ &\qquad B(w_k) \geq d'\xi, \quad \text{and} \\ &\qquad k = 1, \ldots, K, \end{aligned} \tag{8}$$

where ξ = max imum $|\xi_k|$ and $1/d'$ is the minimax approximation error.

From the above formulation of the approximation problem, it is easy to prove that in the optimum solution the following relation always holds:

$$|(G(w_k) - E_k) - R(w_k)| \leq \frac{\xi}{B(w_k)} \leq \frac{1}{d'}. \tag{9}$$

Therefore, if the positive quantity $1/d'$ takes a sufficiently small value, then the fitting error is satisfactory.

Problem (8) is converted to a linear one by using the following transformations:

$$\begin{aligned} \xi &= 1/\xi', & d &= 1/d', \\ a'_m &= a_m/\xi', & b'_m &= b_m/\xi'. \end{aligned} \tag{10}$$

Finally, the approximation problem takes the following linear form:

$$
\begin{aligned}
&\text{minimize } d \\
&\text{subject to:} \\
&+(G(w_k) - E_k)\xi + (G(w_k) - E_k) \sum_{m=1}^{M} b'_m w_k^m - \sum_{m=0}^{N} a'_m w_k^m \le 1 \\
&-(G(w_k) - E_k)\xi - (G(w_k) - E_k) \sum_{m=1}^{M} b'_m w_k^m + \sum_{m=0}^{N} a'_m w_k^m \le 1 \\
&\qquad -\xi - \sum_{m=1}^{M} b'_m w_k^m + d \le 0, \quad \text{and} \\
&\qquad k = 1, \ldots, K.
\end{aligned}
\tag{11}
$$

The solution of the linear problem (11) is obtained by using the revised simplex algorithm and the final value of d represents the optimal minimax approximation error. According to Eq. (4), the optimal values of coefficients $a_m$and b_m form the approximated rational function. Clearly, the initial specifications of the linear programming problem (11) are the number of numerator and denominator coefficients (the variables N and M) and the values of E_k (usually we take $E_k = E$, $\forall k$, where E is a small positive value).

It is noted that in contrast to other similar approximation techniques, such as the Differential Correction Algorithm [17], the LRA algorithm is not iterative; i.e., the linear programming problem is not solved recursively. The only iterations (loops) needed are those included in the simplex algorithm. Although, if we want to adjust the values of M and N in an adaptive scheme, then we must solve iteratively the linear programming approximation problem until, for example, the approximation error becomes sufficiently small. This case is unusual. As we can see in the given examples, small predefined values for N and M result to satisfactory approximation.

The rational function $R(w)$ is real and continuous. To find its minimum in the region $[W_n, W_{n+1}]$, the one-dimensional Golden search algorithm is applied. The inputs of the Golden search algorithm are only the limits of the interval. In our case, the limits are defined by the W_n and W_{n+1}, while the one-dimensional function for each region is the real rational function $R(w)$. To ensure that the Golden search algorithm always converges to the global minimum, we use the following procedure.

Step 1. Find the minimum $R_{\min}$ of $R(w)$.
Step 2. Define the function $Y(w)$ according to the relation

$$
Y(w) = \begin{cases} R(w), & \text{if } R(w) \le R_{\min}, \\ R_{min}, & \text{otherwise.} \end{cases} \tag{12}
$$

Step 3. Find the minimum $Y_{\min}$ of $Y(w)$.
Step 4. If $Y_{\min} = R_{\min}$, go to Step 5; otherwise put $R_{\min} = Y_{\min}$ and go to Step 2.
Step 5. The global minimum solution is equal to $R_{\min}$. Terminate the procedure.

The result of the above minimization procedure is the minimum value of $R(w)$ and its position. This minimum is taken as a threshold value of the histogram.

4. EXPERIMENTAL RESULTS

The proposed multilevel threshold selection method was tested using a variety of digital images. In order to have some comparative results with other existing similar techniques, we have implemented the methods of Otsu, Kapur *et al.*, and Reddi *et al.* It is noted that the results given by the LRA algorithm are referred to normalized histogram values, that is, 0–1 for the x-axis and 0–1 for the y-axis. Here we present three characteristic examples that describe the application and the effectiveness of the proposed method to three images.

Example 1

In the first example, the method was applied to the badly illuminated text image that is shown in Fig. 2. Because this

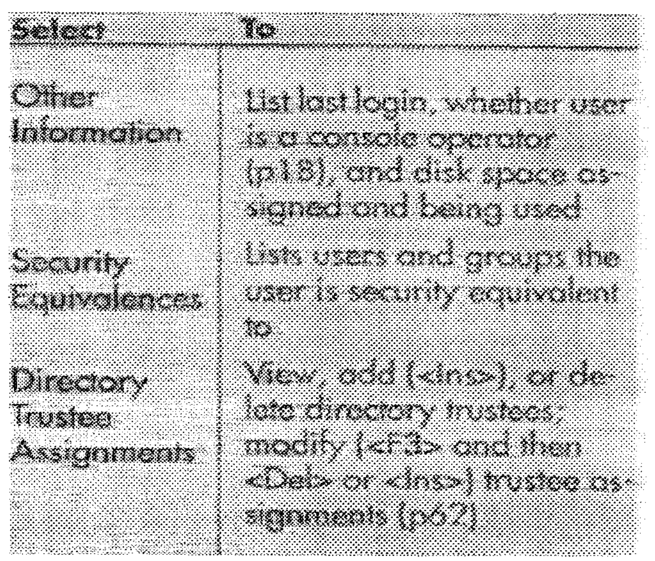

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FIG. 2. Text image for Example 1.

TABLE 1
Histogram Values for Example 1

i	*h(i)*	*i*	*h(i)*	*i*	*h(i)*	*i*	*h(i)*	*i*	*h(i)*	*i*	*h(i)*	*i*	*h(i)*	*i*	*h(i)*
1	2	33	1762	65	356	97	145	129	235	161	0	193	0	225	0
2	11	34	3712	66	0	98	326	130	0	162	0	194	0	226	0
3	81	35	5854	67	366	99	166	131	211	163	637	195	0	227	0
4	0	36	1985	68	310	100	0	132	250	164	626	196	0	228	0
5	170	37	6138	69	579	101	0	133	263	165	0	197	0	229	0
6	631	38	4193	70	538	102	175	134	258	166	608	198	0	230	0
7	354	39	0	71	0	103	157	135	0	167	499	199	0	231	0
8	417	40	2150	72	459	104	143	136	279	168	0	200	0	232	0
9	163	41	6447	73	0	105	361	137	0	169	0	201	0	233	0
10	201	42	4359	74	197	106	184	138	289	170	0	202	0	234	0
11	0	43	2138	75	425	107	0	139	293	171	0	203	0	235	0
12	97	44	6502	76	194	108	0	140	305	172	372	204	0	236	0
13	104	45	4124	77	200	109	203	141	296	173	229	205	0	237	0
14	235	46	0	78	182	110	185	142	0	174	0	206	0	238	0
15	180	47	2058	79	363	111	173	143	0	175	136	207	0	239	0
16	367	48	3950	80	0	112	179	144	310	176	0	208	0	240	0
17	523	49	3604	81	174	113	150	145	379	177	0	209	0	241	0
18	0	50	3288	82	179	114	0	146	0	178	91	210	0	242	0
19	478	51	3040	83	185	115	0	147	377	179	0	211	0	243	0
20	581	52	0	84	183	116	189	148	0	180	42	212	0	244	0
21	755	53	2638	85	330	117	392	149	0	181	0	213	0	245	0
22	1493	54	1235	86	178	118	209	150	363	182	0	214	0	246	0
23	1186	55	2164	87	0	119	199	151	378	183	0	215	0	247	0
24	1540	56	1886	88	0	120	164	152	409	184	0	216	0	248	0
25	0	57	855	89	333	121	0	153	402	185	14	217	0	249	0
26	2776	58	1592	90	153	122	0	154	0	186	0	218	0	250	0
27	1050	59	0	91	153	123	228	155	0	187	5	219	0	251	0
28	3595	60	1290	92	164	124	198	156	492	188	0	220	0	252	0
29	2763	61	1201	93	158	125	196	157	0	189	0	221	0	253	0
30	2963	62	0	94	0	126	236	158	0	190	0	222	0	254	0
31	4973	63	948	95	154	127	225	159	544	191	0	223	0	255	0
32	0	64	837	96	157	128	0	160	536	192	0	224	0	256	0

image includes only text, it is obvious that the suitable number of histogram peaks is equal to 2. Table 1 gives the histogram values.

Figure 3 describes the results obtained from the application of the new method. Specifically, the hill-clustering algorithm converges to the 39 and 151 peak locations. The histogram of this image is approximated in the interval [39, 151] by a rational function $R(w)$ with $M = 4$, $N = 2$,

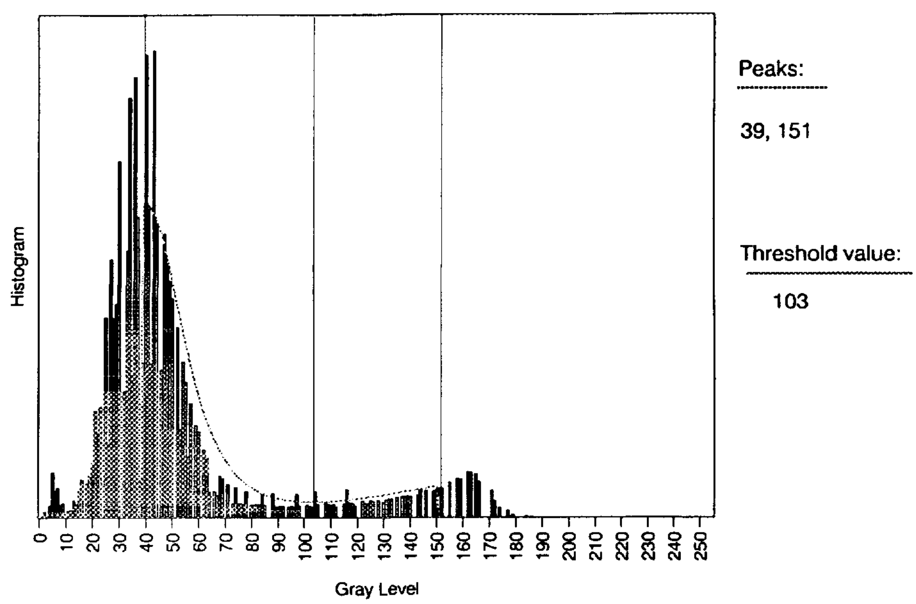

FIG. 3. Application of the new method to the image of Fig. 2.

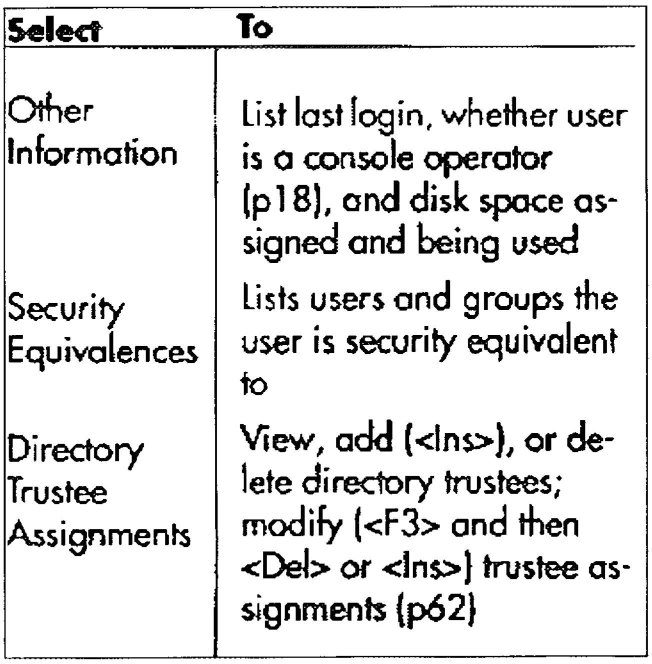

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FIG. 4. Image of Fig. 2 after global thresholding.

and $E = 0.0$. The solution of the linear minimax approximation problem (11) gives the following optimal values:

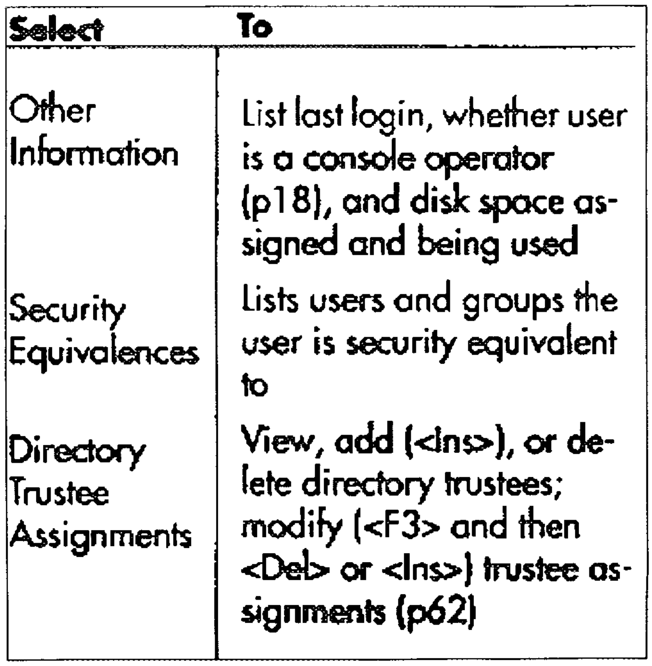
$$\xi = 0.0554956, \quad \delta = 0.338864,$$
$$a_1 = 0.107191, \quad b_1 = -9.80346,$$
$$a_2 = 0.65468, \quad b_2 = 28.729.$$
$$a_3 = -5.2217,$$
$$a_4 = 8.17153,$$

As we can observe from the above results and from Fig. 3, the histogram fitting is satisfactory between the peaks and the rational function has only one minimum. By using the Golden search technique, the location of this minimum was found to correspond to the 103 gray level, which is taken as the threshold value. Therefore, according to relation (2), the final binary image has been obtained and it has the form of Fig. 4.

For comparison, we apply to the same histogram, the methods of Otsu, Kapur *et al.*, and Reddy *et al.* Otsu's method, using the maximum $\{\sigma_B^2/\sigma_T^2\}$ criterion, converges for a threshold equal to 86. Additionally, the method of Kapur *et al.*, based on the maximum entropy criterion, gives a threshold value equal to 64, which corresponds to maximum entropy equal to 7.821075. Finally, the method of Reddi *et al.*, after four iterations, converges to a threshold value equal to 87. Figure 5 shows the segmented images resulted by using the methods of Otsu and Kapur *et al.*

Example 2

As a second example, let us consider the application of the new multilevel threshold method to the image of Fig. 6. This image consists of five colors and the back-



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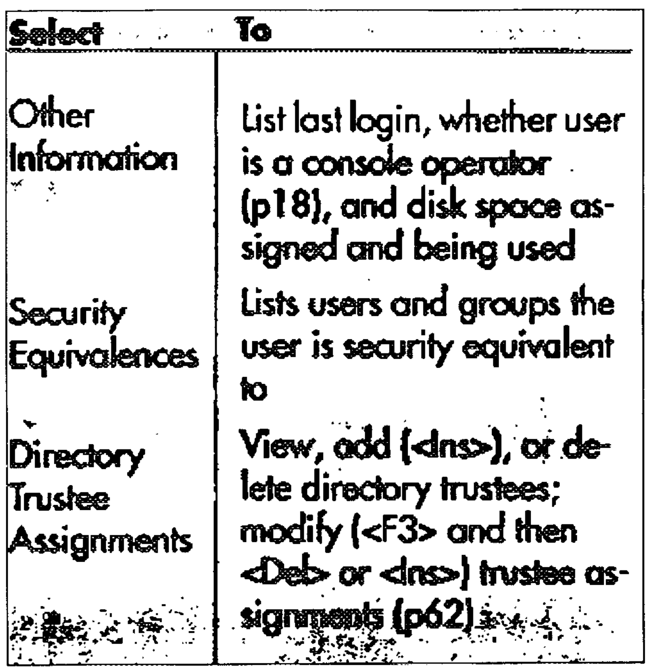
(a)



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(b)

FIG. 5. Thresholding by using the methods of (a) Otsu [11] and (b) Kapur *et al.*[9].

FIG. 6. Image for Example 2.

FIG. 8. The final segmented image of Fig. 6 with only four gray levels.

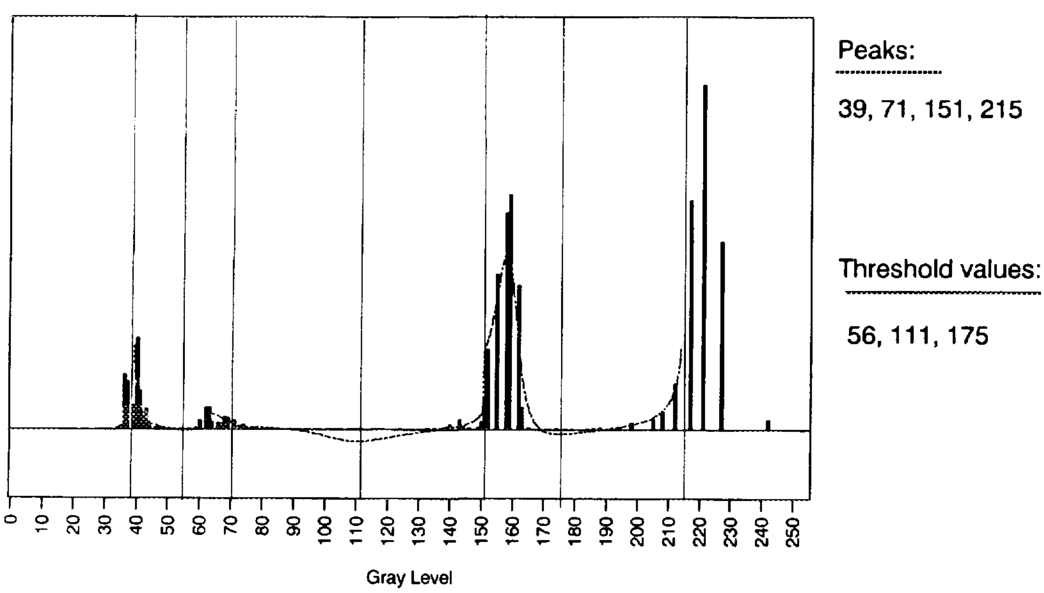

FIG. 7. Application of the new method to the image of Fig. 5.

TABLE 2
Histogram Values for Example 2

i	h(i)	i	h(i)	i	h(i)	i	h(i)	i	h(i)	i	h(i)	i	h(i)	i	h(i)
1	0	33	0	65	550	97	22	129	37	161	0	193	138	225	0
2	0	34	0	66	0	98	66	130	0	162	0	194	0	226	0
3	0	35	137	67	514	99	34	131	48	163	10564	195	202	227	0
4	0	36	290	68	440	100	0	132	40	164	1618	196	0	228	13696
5	0	37	4002	69	902	101	0	133	41	165	0	197	0	229	0
6	0	38	3522	70	867	102	28	134	76	166	124	198	0	230	0
7	0	39	0	71	0	103	29	135	0	167	61	199	490	231	0
8	0	40	1770	72	709	104	28	136	77	168	0	200	0	232	0
9	0	41	6681	73	0	105	64	137	0	169	0	201	0	233	0
10	0	42	2828	74	268	106	29	138	96	170	0	202	0	234	0
11	0	43	1027	75	356	107	0	139	136	171	0	203	0	235	0
12	0	44	1481	76	103	108	0	140	206	172	62	204	0	236	0
13	0	45	481	77	69	109	40	141	351	173	49	205	0	237	0
14	0	46	0	78	58	110	29	142	0	174	0	206	719	238	0
15	0	47	147	79	118	111	32	143	0	175	73	207	0	239	0
16	0	48	199	80	0	112	33	144	681	176	0	208	0	240	0
17	0	49	137	81	44	113	32	145	197	177	0	209	1280	241	0
18	0	50	95	82	37	114	0	146	0	178	56	210	0	242	0
19	0	51	75	83	38	115	0	147	93	179	0	211	0	243	752
20	0	52	0	84	29	116	41	148	0	180	66	212	0	244	0
21	0	53	53	85	68	117	64	149	0	181	0	213	3348	245	0
22	0	54	26	86	40	118	32	150	151	182	0	214	0	246	0
23	0	55	43	87	0	119	32	151	624	183	0	215	0	247	0
24	0	56	25	88	0	120	35	152	2270	184	0	216	0	248	0
25	0	57	9	89	61	121	0	153	5869	185	63	217	0	249	0
26	0	58	46	90	33	122	0	154	0	186	0	218	16682	250	0
27	0	59	0	91	36	123	34	155	0	187	80	219	0	251	0
28	0	60	140	92	30	124	28	156	11332	188	0	220	0	252	0
29	0	61	711	93	34	125	27	157	0	189	122	221	0	253	0
30	0	62	0	94	0	126	35	158	0	190	0	222	25105	254	0
31	0	63	1587	95	33	127	34	159	15743	191	0	223	0	255	0
32	0	64	1618	96	32	128	0	160	17085	192	0	224	0	256	0

ground. The values of the histogram are given in Table 2. Figure 7 shows the approximated results derived for this example by taking four peaks, $N = 3$, $M = 4$, and $E = 0.0$. The three threshold values were found to be equal to 56, 111, and 175, and the final segmented image is shown in Fig. 8. Table 3 gives the approximation errors and the coefficients for the three rational functions.

For comparison, we apply in the same image the multithreshold selection methods of Kapur *et al.* and Reddi *et al.* The method of Kapur *et al.*, in its multithresholding version, tries to find the threshold values by maximizing the total entropy. Generally, for k thresholds $T_1, T_2, \ldots, T_k$, the algorithm maximizes the evaluation function

$$y(T_1, T_2, \ldots, T_k) = -\sum_{i=1}^{T_1} \frac{p_i}{p_{T_1}} \ln\left(\frac{p_i}{p_{T_1}}\right) - \sum_{i=T_1+1}^{T_2} \frac{p_i}{p_{T_2}} \ln\left(\frac{p_i}{p_{T_1}}\right) - \cdots - \sum_{i=T_{k-1}+1}^{T_k} \frac{p_i}{p_{T_k}} \ln\left(\frac{p_i}{p_{T_1}}\right) - \sum_{i=T_k+1}^{n} \frac{p_i}{p_{T_{k+1}}} \ln\left(\frac{p_i}{p_{T_{k+1}}}\right), \tag{13}$$

where

$$p_{T_j} = \sum_{i=T_j+1}^{T_{j+1}} p_i \quad \text{and} \quad p_{T_{k+1}} = \sum_{i=T_k+1}^{n} p_i. \tag{14}$$

For this example, the method of Kapur *et al.* converges for $y = 11.2691$ and for the threshold values 75, 105, and 138. The segmented image in this case is shown in Fig. 9.

The method of Reddi *et al.* starts from the threshold values 86, 138, and 190 and converges to the threshold

TABLE 3
Experimental Results for Example 2

	First rational function	Second rational function	Third rational function
ξ	0.000248	0.01027	0.00131
d	0.056778	0.03637	0.17681
b_1	0.0113275	0.21621	0.37733
b_2	−0.103184	−1.0144	−1.07743
b_3	0.234564	1.13972	0.766917
b_4	−27.595	104.296	50.3737
a_1	258.465	−682.056	−232.963
a_2	−1000	1443.08	342.531
a_3	1375.8	−1000	−164.502

FIG. 9. Application of the Kapur *et al.* [9] method to the image of Fig. 6.

FIG. 10. Application of the Reddi *et al.* [8] method to the image of Fig. 6.

values 54, 112, and 188 after six iterations. Figure 10 shows the segmented image for this case.

Example 3

In the third example, we examine the more complex image of Figure 11. This image contains overlapping objects. The application of the proposed method to this image with three peaks, $N = 4$, $M = 3$, and $E = 0$, results in threshold values equal to 28, 59, and 140. Table 4 gives the approximation errors and the coefficients of the rational polynomial. Figure 12 depicts the histogram and its approximation by the rational functions, whereas Fig. 13 shows the final segmented image.

We apply, as in the previous examples, the methods of Kapur *et al.* and the method of Reddy *et al.* For the image of this example, the method of Kapur gives threshold values equal to 46, 91, and 136 that correspond to $y(46, 91, 136) = 13.4732$. Figure 14 shows the segmented image. Also, the method of Reddi *et al.* results in the threshold values 50, 106, and 158 after four iterations and gives the segmented image of Fig. 15.

We will note that the least-squares algorithm can substitute for the rational approximation method as the approximation procedure. However, there are some significant "quality" differences and disadvantages using the least-

TABLE 4
Experimental Results for Example 3

	First rational function	Second rational function	Third rational function
ξ	0.168475	0.001464	0.000758
d	0.168491	0.14871	0.011734
b_1	0.596715	0.05355	−0.06118
b_2	27.6382	−0.62694	0.388558
b_3	−622.309	2.39339	−0.797522
a_1	2967.02	−2.87871	0.544024
a_2	−0.00451	−14.9706	−5.75590
a_3	0.06731	70.7315	11.4808
a_4	−0.28732	−99.0335	−7.33152

FIG. 11. Image for Example 3.

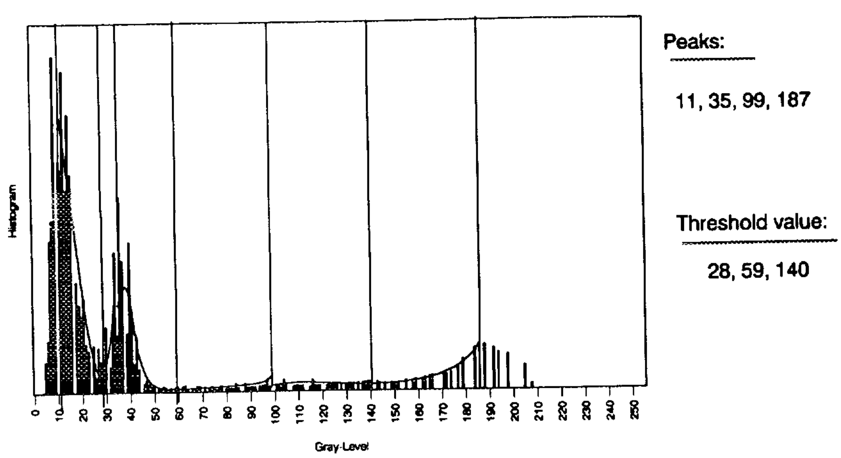

FIG. 12. Histogram and approximation results for Example 3.

FIG. 13. Segmented image of Example 3.

squares method. Specifically, the rational approximation technique is less sensitive to noise because it minimizes the maximum absolute differences instead of the sum of the squared differences. Also, clearly a rational polynomial can more efficiently fit difficult data because it has numerator and denominator. Therefore, for equivalently fitting results, rational functions need fewer coefficients than do polynomials. Additionally, the coefficients derived from the least-squares method are unstable in magnitude if the order of the polynomial exits an upper bound.

5. CONCLUSIONS

This paper introduces a new methodology for multilevel threshold selection. The proposed approach consists of three main steps. Initially, a hill-clustering method is used to approximately determine the histogram peaks. Next, a fast linear rational approximation algorithm is applied and the histogram segments are approximated by real rational functions according to the minimax criterion. Finally, the one-dimensional Golden search method finds the global minimum values of the rational functions. These minima are specified as the multilevel threshold values of the histogram.

Experimental results show that for certain types of images the proposed method produces satisfactory multithresholding results. The proposed technique was compared, using experimental results, with other well-known approaches and specifically, with the methods of Otsu [11], Kapur *et al.* [4], and Reddi *et al.* [8]. In comparison with the above techniques, the new multithresholding approach has the main qualitative difference that it is based on and uses the morphology of the histogram and it is not only a statistical procedure. In Example 2, it is obvious

FIG. 14. Segmented image by using the method of Kapur *et al.* [9].

that we have three well-discriminated valleys. However, the method of Kapur *et al.* fails because it results in thresholds belonging to only one valley. In the same example, the proposed approach and the method of Reddi *et al.* give similar results. In Example 3, both the methods of Kapur *et al.* and Reddi *et al.* fail because they do not give a threshold value in the first valley. Additionally, in comparison to the least-squares algorithm, the linear rational approximation technique is more effective and gives better approximation results.

APPENDIX: NOMENCLATURE

$H(k)$	image histogram
$f(x, y)$	image function: gives the gray-level value of the image at the (x, y) pixel
$C(k)$	total number of pixels that have gray-level values equal to k
$I(m)$	the mth thresholding value
L	the total number of threshold values
$F(x, y)$	the image after multilevel thresholding
M_o	the maximum number of desired histogram peaks
f_i	the gray-level i of the histogram
$g_{i,ITER}$	the frequency of the cell i
d_i	arrow direction at cell i
P	total number of histogram peaks
K	total number of gray levels in a specific histogram segment
$G(w_n)$	the normalized frequencies of the histogram
w_k	the kth gray-level value
W_n	limit of a histogram segment
$R(w)$	rational function
$A(w)$	numerator of the rational function
$B(w)$	denominator of the rational function
a_m	coefficients of $A(w)$

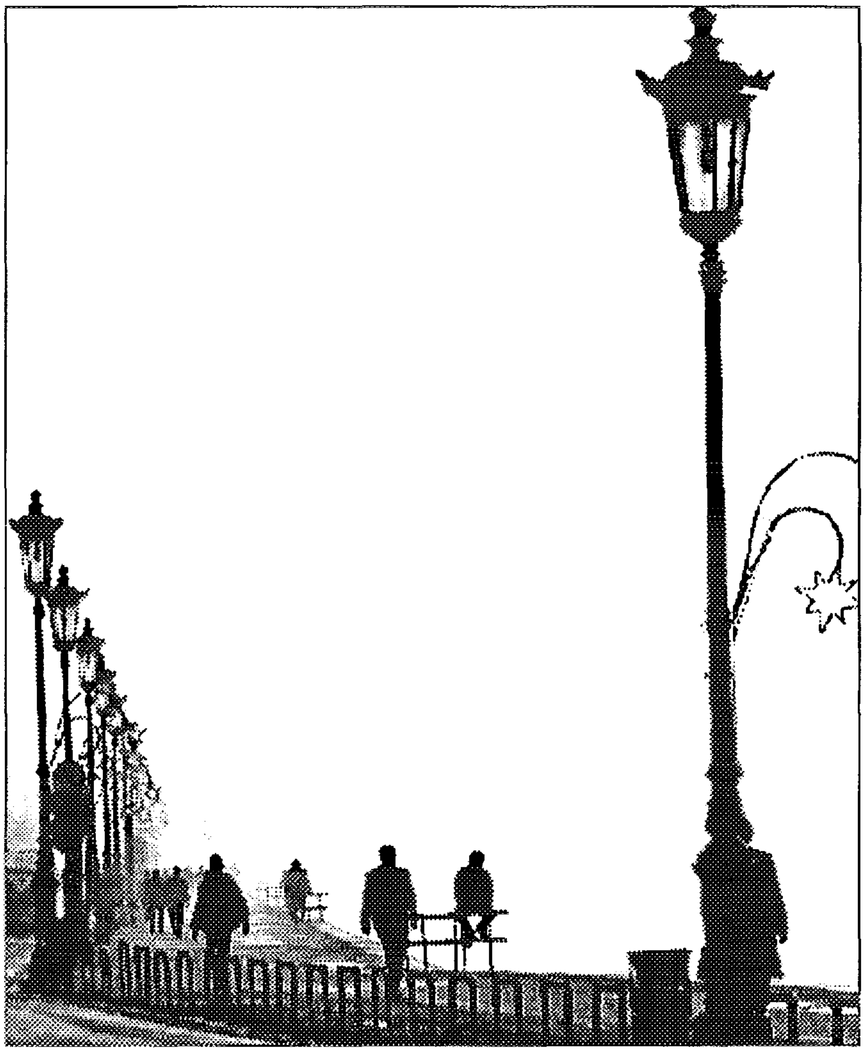

FIG. 15. Segmented image by using the method of Reddi *et al.* [8].

b_m	coefficients of $B(w)$
E_k	auxiliary variables with small absolute values
ξ	auxiliary variable of the rational approximation problem
δ	the minimax approximation error
N_C	number of cells in each iteration of the hill-clustering algorithm

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